A New Generalized Form of the Exponential Distribution: Properties and Application

by

Dr. Mohamed Al-Sayed Mead

Professor in the Department of Statistics and Insurance
Faculty of Commerce, Zagazig University
mead9990@gmail.com

Dr. Ahmed Zakaria Muhammad Afify

An assistant professor in the Department of Statistics mathematics and Insurance, Faculty of Commerce, Banha University
ahmed.afify@fcom.bu.edu.eg

Manar Ibrahim Mohammed Basyouny

Master's researcher in the Department of Statistics and Insurance, Faculty of Commerce, Zagazig University
manarbassiouni74@gmail.com

Journal of Business Research
Faculty of Commerce -Zagazig University
Volume 47 - Issue 4 October 2025
link: https://zcom.journals.ekb.eg/

Abstract

In this article, we propose a new extension of the exponential distribution. We investigate some of their statistical properties, including moments, quantile and moment generating functions, emphasizing their utility in modeling diverse aging and failure criteria. One key advantage of the proposed distribution lies in its capacity to represent its density as a mixture of exponential densities, offering both symmetric and asymmetric shapes for greater modeling flexibility. The estimation of the model parameters is achieved through maximum likelihood estimation. The study presents comprehensive simulation results to assess the effectiveness of the proposed estimation technique. Furthermore, a practical application on real-world data is conducted to showcase the adaptability and versatility of the introduced distribution when compared with other extensions of the exponential model.

Keywords: Maximum likelihood estimation; Exponential distribution; Simulations; Data analysis .

1. Introduction

Recently, many generalized families of distributions have been proposed and extensively used in modeling data in various applied sciences such as economics, finance, insurance, engineering and life testing. However, there is a clear need for extended forms of these distributions by adding one or more shape parameter(s) in order to obtain greater flexibility in modelling these data. So, several classes of distributions have been constructed by extending common families of continuous distributions. Gupta et al. (1998) who proposed the exponentiated-G class, which consists of raising the cumulative distribution function (CDF) to a positive power parameter. The idea of beta-generated (B-G) family of distributions stemmed from the paper of Eugene et al. (2002), Generalized beta distributions are widely studied in statistics and numerous authors have developed various classes of these distributions. Some other beta-generated families are also been discussed in the literature. Some others families are reobtained by means of the maximum entropy principle by Zografos (2008) who considered the beta-Weibull distribution through this principle. Pescim et al. (2010) and Paranaıba et al. (2011) have studied important mathematical properties of the beta generalized half-normal and beta Burr XII distributions. Many other classes can be cited such as Kumaraswamy-G by Cordeiro and de Castro (2011), exponentiated generalized-G by Cordeiro et al. (2013), beta Marshall-Olkin-G by Alizadeh et al. (2015), beta Weibull-G family by Yousof et al. (2017), beta transmuted-H families by Afify et al. (2017), beta Nadarajah-Haghighi distribution by Dias et al. (2018), the modified beta transmuted family by Awodutire et al. (2021), odd

Fréchet-G family by Sadiq et al. (2023) and alpha–beta-power family by Semary et al. (2024).

In this paper, we introduce a new four parameter beta alpha power exponential (BAPEx) distribution, the exponential (Ex) distribution has been extensively used in analyzing lifetime data due to its lack of memory property and its simple form. However, the Ex-distribution with only a constant hazard rate shape is not able to fit data sets with different hazard shapes as increasing, decreasing, bathtub, or unimodal (upside down bathtub) shaped failure rates, often encountered in engineering and reliability, among others.

The rest of the paper is organized into seven sections. In Section 2 We define the new BAP-H family and one of special sub-models the BAPEx model. In Section 3, we derive a linear representation for the BAP-H density function. In section 4, we obtain some of its statistical properties. Section 5, Inference about the BAPEx distribution parameters is presented. Section 6 provides a simulation study. A real-life data application is presented in Section 7. Section 8 gives some conclusions.

2. The proposed Family

Alpha power transformation-H (APT-H) family was proposed by Mahdavi and Kundu (2017). Consider a baseline CDF $G(x;\alpha)$ with corresponding probability densty function (PDF) $g(x;\alpha)$ and parameter α. Then, the CDF of the APT-H family (for $x \in R$) has the form

$$G(x) = \frac{\alpha^{H(x)}-1}{\alpha-1}, \qquad \textbf{(1)}$$

The corresponding PDF of the APT-H class is given by

$$g(x) = \frac{log(\alpha)}{\alpha-1} h(x)\alpha^{H(x)}, \quad \textbf{(2)}$$

where $\alpha > 0$, $\alpha \neq 1$. For α= 1, Equation (2) reduces to the baseline distribution. Further details can be found in Mahdavi and Kundu (2017). The B-G family was defined by Eugene et al. (2002). The CDF of the B-G family (for x ∈ R) has the form

$$F(x) = \frac{1}{\beta(\delta,\eta)} \int_0^{G(x)} t^{(\delta-1)}(1-t)^{\eta-1}dt = I_{G(x)}(\delta,\eta). \quad \textbf{(3)}$$

The corresponding PDF of the B-G class is given by

$$f(x) = \frac{1}{\beta(\delta,\eta)} \quad g(x) \quad G(x)^{\delta-1}[1-G(x)]^{\eta-1}, \quad \textbf{(4)}$$

where $\delta > 0$ and $\eta > 0$ are two additional shape parameters, $Iy(\delta,\eta) = \beta y(\delta,\eta)/\beta(\delta,\eta)$ is the incomplete beta function ratio.

By using Mahdavi and Kundu (2017) and Eugean et al. (2002) then the BAP-H family which CDF (for $x \in R$) is given by

$$F(x) = \frac{1}{\beta(\delta,\eta)} \int_0^{\frac{\alpha^{H(x)}-1}{\alpha-1}} t^{(\delta-1)}(1-t)^{\eta-1}dt = I_{\frac{\alpha^{H(x)}-1}{\alpha-1}}(\delta,\eta). \quad \textbf{(5)}$$

The corresponding PDF of the BAP-H family is given by

$$f(x) = \frac{1}{\beta(\delta,\eta)} \quad \frac{log(\alpha)}{\alpha-1} h(x)\alpha^{H(x)} \quad \left[\frac{\alpha^{H(x)}-1}{\alpha-1}\right]^{\delta-1} \left[1-\frac{\alpha^{H(x)}-1}{\alpha-1}\right]^{\eta-1}, \quad \textbf{(6)}$$

where $\delta > 0, \eta > 0$ and $\alpha > 0, \alpha \neq 1$.

For $\alpha = 1$, we get the B-G family $H(x)$. A random variable X having the PDF (6)will be denoted by X ~BAP-H (δ, η, α). BAP-H family and its Sub-families are introduced in Table 1.

Table 1: BAP-H family and its Sub-families.

BAP-H family	δ	η	α	New
B-G family	δ	η	1	Eugene et al. (2002)
APT-H family	1	1	α	Mahdavi and Kundu (2017)
exp-G family	δ	1	1	Gupta et al. (1998)

The Ex distribution with scale parameter $\lambda > 0$ has PDF and CDF given by $h(x) = \lambda e^{-\lambda x}$ (for $x > 0$) and $H(x) = 1 - e^{-\lambda x}$, respectively. Then, the CDF of the BAPEx distribution (for $x > 0$) has the form

$$F(x) = \frac{1}{\beta(\delta,\eta)} \int_0^{\frac{\alpha^{1-e^{-\lambda x}}-1}{\alpha-1}} t^{(\delta-1)}(1-t)^{\eta-1} dt = I_{\frac{\alpha^{1-e^{-\lambda x}}-1}{\alpha-1}}(\delta,\eta). \qquad \textbf{(7)}$$

The corresponding PDF of the BAPEx distribution is given by

$$f(x) = \frac{1}{\beta(\delta,\eta)} \frac{\log(\alpha)}{\alpha-1} \lambda e^{-\lambda x} \alpha^{1-e^{-\lambda x}} \left[\frac{\alpha^{1-e^{-\lambda x}}-1}{\alpha-1}\right]^{\delta-1} \left[1-\frac{\alpha^{1-e^{-\lambda x}}-1}{\alpha-1}\right]^{\eta-1} \qquad \textbf{(8)}$$

For $\alpha = 1$, we get the BEx distribution. A random variable X having the PDF (8) will be denoted by $X \sim$BAPEx $(\delta, \eta, \alpha, \lambda)$. The quantile function of the BAPEx , $Q(u) = F^{-1}(p)$, can be obtained by inverting (7) numerically.

Figure 1 presents the PDF of the BAPEx distribution for different parameter values. The plots demonstrate the flexibility of the model in capturing various shapes, including skewed and symmetric behaviors, which makes it suitable for modeling diverse types of data. Figure 2 shows the HRF of the BAPEx distribution for several parameter combinations. The results indicate that the distribution can accommodate increasing, decreasing, and bathtub-shaped hazard functions, emphasizing its applicability in reliability and survival analysis.

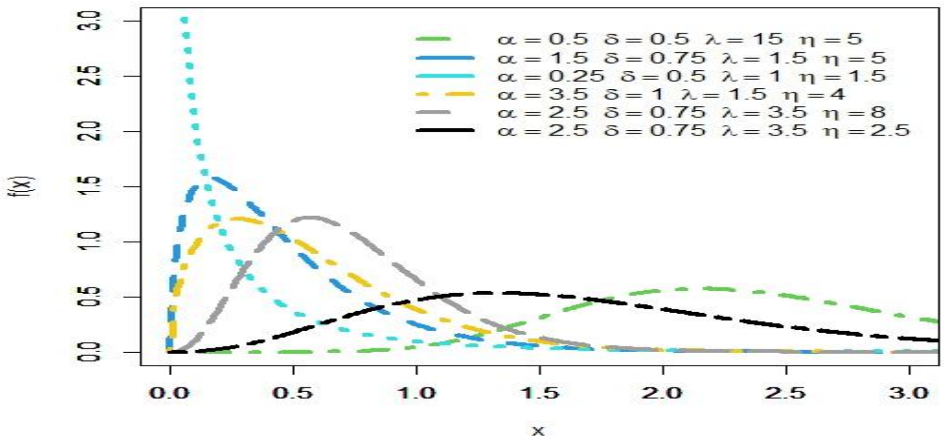

Figure 1: Plots of the PDF of the BAPEx distribution with different parameter values.

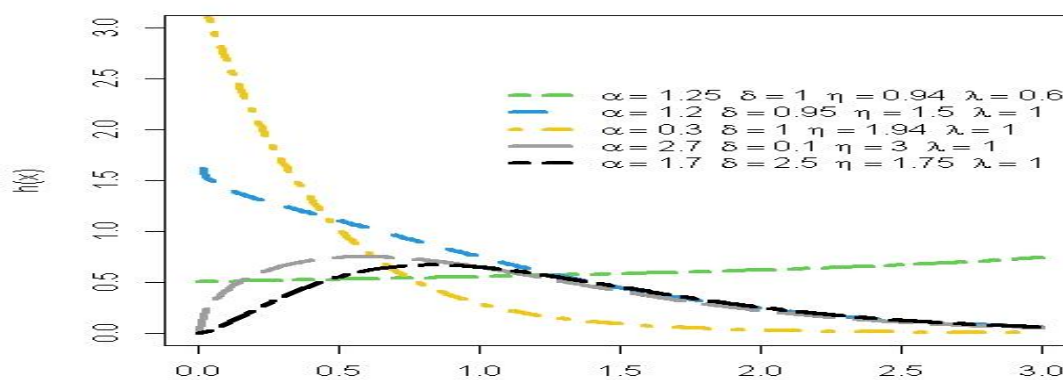

Figure 2: Plots of the HRF of the BAPEx distribution with different parameter values.

3. Mixture Representation

In this section, we derive a useful representation for the BAP-H family density. The derived representation is crucial for the derivation of mathematical properties. It allows moments, moment generating functions (MGF), order statistic properties, etc., to be expressed as mixtures of BAP-H family. Using the following power series

$$(1-z)^q = \sum_{k=0}^{\infty} \binom{q}{k} (-1)^k z^k, \quad |z| > 0 \; and \;\; q > 0. \tag{9}$$

A useful mixture represenation of the BAP-H density in (6) follows as

$$f(x) = \frac{\log(\alpha)}{\beta(\delta,\eta)} \quad h(x)\alpha^{H(x)} \sum_{i=0}^{\infty} \frac{(-1)^i}{(\alpha-1)^{i+\delta}} \binom{\eta-1}{i} \left[\alpha^{H(x)}-1\right]^{i+\delta-1}.$$

Hence

$[\alpha^{H(x)}-1]^{i+\delta-1} = \alpha^{(i+\delta-1)H(x)}[1-\alpha^{-H(x)}]^{i+\delta-1}.$

Applying the power series (7) in th following terme

$$\alpha^{(i+\delta-1)H(x)}[1-\alpha^{-H(x)}]^{i+\delta-1}$$

$$= \alpha^{(i+\delta-1)H(x)} \sum_{j=0}^{\infty} (-1)^j \binom{i+\delta-1}{j} \alpha^{-jH(x)}$$

$$= \sum_{j=0}^{\infty} (-1)^j \binom{i+\delta-1}{j} \alpha^{(i+\delta-j-1)H(x)}.$$

Then, we can write

$$f(x) = \frac{\log(\alpha)\, h(x)}{\beta(\delta,\eta)} \quad \sum_{i,j=0}^{\infty} \binom{\eta-1}{i} \binom{i+\delta-1}{j} \frac{(-1)^{i+j}}{(\alpha-1)^{i+\delta}} \alpha^{[i+\delta-j]H(x)}.$$

Using the follow power series

$$\alpha^q = \sum_{k=0}^{\infty} \frac{(\log \alpha)^k}{k!} q^k, \qquad q > 0.$$

Hence

$$\alpha^{(i+\delta-j)H(x)} = \sum_{k=0}^{\infty} \frac{(\log \alpha)^k}{k!} (i+\delta-j)^k H(x)^k.$$

The PDF of the BAP-H class reduces to

$$f(x) = \frac{h(x)}{\beta(\delta,\eta)} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j} [\log(\alpha)]^{k+1} (i+\delta-j)^k}{k!\,(\alpha-1)^{i+\delta}}$$

$$\binom{\eta-1}{i}\binom{i+\delta-1}{j} H(x)^k.$$

Then, we have

$$f(x) = \sum_{k=0}^{\infty} b_k g_{k+1}(x)\,, \qquad (\mathbf{10})$$

where

$$b_k = \frac{1}{\beta(\delta,\eta)} \sum_{i,j=0}^{\infty} \binom{\eta-1}{i}\binom{i+\delta-1}{j} \frac{(-1)^{i+j} [\log(\alpha)]^{k+1} (i+\delta-j)^k}{(k+1)!\,(\alpha-1)^{i+\delta}}$$

and

$$g_{k+1}(x) = (k+1) h(x) H(x)^k.$$

$g_{k+1}(x)$ is the exponentiated-H (exp-H) with power parameter $(k+1)$. Thus, several mathematical properties of the BAP-H family can be obtained simply from those of the exp-H family. Equation (10) is the main result of this section.

The CDF of the BAP-H family can also be expressed as a mixture of exp-H CDFs. By integrating (10) , we obtain the mixture representation

$$F(x) = \sum_{k=0}^{\infty} b_k G_{k+1}(x), \qquad (\mathbf{11})$$

where $G_{k+1}(x) = H(x)^{k+1}$ is the CDF of the exp-H with power parameter $k + 1$.

4 Statistical Properties

In this section, the main statistical properties of the proposed distribution family are presented. These include the derivation of the moments and the moment generating function (MGF), as well as the analysis of order statistics. Furthermore, the entropy measure is discussed to explore the uncertainty associated with the distribution. Finally, the quantile function is derived, which plays a key role in simulation and statistical

4.1 Moment

The r-th moment of X, denoted by μ_r', is given by:

$$\mu_r' = E(X^r) = \sum_{k=0}^{\infty} b_k E(Y_{k+1}^r),$$

where Y_{k+1} is an exp-H random variable with power parameter $k + 1$.The n-th central moment of X, denoted by μ_n, is given by:

$$\mu_n = E[(X - \mu_1')^n] = \sum_{r=0}^{n} \binom{n}{r} (-\mu_1')^{n-r} E(X^r).$$

4.2 Moment Generating Function

In probability theory and statistics, the MGF of a real-valued random variable is an alternative specification of its probability distribution. Thus, it provides the basis of an alternative route to analytical results compared with working directly with probability density functions or cumulative distribution functions. There are particularly simple results for the MGF of distributions defined by the weighted sums of random variables. However, not all random variables have MGF, see Hogg et al. (2005).

The MGF $M_X(t) = E(e^{tX})$ is given by:

$$M_X(t) = \sum_{k=0}^{\infty} b_k M_{k+1}(t),$$

where $M_{k+1}(t)$ is the MGF of $\mathrm{Y_{k+1}}$.

4.3 Order Statistics

The order statistics are important in statistical theory. Let $X_1, \dots, X_n$ be a random sample from the BAP-H family. The PDF of the i-th order statistic, $X_{i:n}$, is

$$f_{X_{i:n}}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{j+i-1}. \quad (12)$$

The term $F_{X_{i:n}}(x)^{j+i-1}$ can be expressed as:

$$F_{X_{i:n}}(x)^{j+i-1} = \left[\sum_{k=0}^{\infty} b_k H(x)^{k+1}\right]^{j+i-1}. \quad (\mathbf{13})$$

Using the identity $(\sum_{r=0}^{\infty} b_r u^r)^n = \sum_{r=0}^{\infty} C_{n,r} u^r$, (see Gradshteyn and Ryzhik, 2014) where the coefficients $C_{n,r}$ are easily determined from the recurrence equation , we obtain

$$F_{X_{i:n}}(x)^{j+i-1} = \left\{\sum_{k=0}^{\infty} b_k \left[H(x)^{\frac{(k+1)}{k}}\right]^k\right\}^{j+i-1}$$

$$= \sum_{k=0}^{\infty} C_{k,i+j-1} \left[H(x)^{\frac{(k+1)}{k}}\right]^k \quad . \qquad \textbf{(14)}$$

Substituting (10) and (14) into (12) and using a power series expansion, the PDF of $X_{i:n}$ can be expressed as

$$f_{X_{i:n}}(x) = \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \frac{1}{B(i, n-i+1)} \sum_{k=0}^{\infty} C_{k,i+j-1} h_{k+1}(x),$$

where $h_{\mathrm{k+1}}(x)$ is the exp-H PDF with power parameter $k+1$.
It follows that the PDF of BAP-H order statistics is a mixture of exp-H PDFs. Hence, the properties of $X_{i:n}$ follow from the properties of X_{k+1}.

4.4 Entropy

The Rényi entropy of a random variable X is defined by:

$$I_\theta(X) = \frac{1}{1-\theta} log\left(\int_{-\infty}^{\infty} f(x)^\theta \, dx\right), \quad \theta > 0, \quad \theta \neq 1.$$

$$f(x)^{\theta} = \left\{ \frac{1}{\beta(\delta,\eta)} \quad \frac{\log(\alpha)}{\alpha-1} h(x) \alpha^{H(x)} \quad \left[\frac{\alpha^{H(x)}-1}{\alpha-1}\right]^{\delta-1} \left[1 - \frac{\alpha^{H(x)}-1}{\alpha-1}\right]^{\eta-1} \right\}^{\theta}.$$

Applying the power series (9) in th following equation

$$f(x)^{\theta} = \left[\frac{1}{\beta(\delta,\eta)}\right]^{\theta} \left(\frac{\log(\alpha)}{\alpha-1}\right)^{\theta} h(x)^{\theta} \alpha^{\theta\, H(x)} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(\alpha-1)^{k+\delta\theta-\theta}} \binom{\theta(\eta-1)}{k} \left[\alpha^{H(x)}-1\right]^{k+\theta\delta-\theta}.$$

Hence

$$[\alpha^{H(x)}-1]^{k+\delta\theta-\theta} = \alpha^{(k+\delta\theta-\theta)H(x)}[1-\alpha^{-H(x)}]^{k+\delta\theta-\theta}.$$

Applying the power series (9) in th following terme

$$\alpha^{(k+\delta\theta-\theta)H(x)}[1-\alpha^{-H(x)}]^{k+\delta\theta-\theta} = \sum_{j=0}^{\infty} (-1)^{j} \binom{k+\delta\theta-\theta}{j} \alpha^{(k+\delta\theta-j-\theta)H(x)}.$$

Then, we can write

$$f(x)^{\theta} = \left[\frac{\log(\alpha)\, h(x)}{\beta(\delta,\eta)}\right]^{\theta} \sum_{k,j=0}^{\infty} \binom{\theta(\eta-1)}{k} \binom{k+\delta\theta-\theta}{j} \frac{(-1)^{i+j}}{(\alpha-1)^{k+\delta\theta}} \alpha^{[k+\delta\theta-j]H(x)}.$$

Using the follow power series

$$\alpha^{q} = \sum_{k=0}^{\infty} \frac{(\log \alpha)^{\mathrm{k}}}{k!} q^{k}, \qquad q > 0.$$

Hence

$$\alpha^{(k+\delta\theta-j)H(x)} = \sum_{m=0}^{\infty} \frac{(\log \alpha)^m}{m!} (k+\delta\theta-j)^m H(x)^m.$$

The PDF of the BAP-H class reduces to

$$f(x)^{\theta} = \left[\frac{h(x)}{\beta(\delta,\eta)}\right]^{\theta}$$

$$\sum_{k,j,m=0}^{\infty} \frac{(-1)^{k+j}[\log(\alpha)]^{m+\theta}(k+\delta\theta-j)^m}{k!\,(\alpha-1)^{k+\delta\theta}} \binom{\theta(\eta-1)}{k} \binom{k+\delta\theta-\theta}{j} H(x)^m.$$

The Rényi entropy of the BAP-H family is given by

$$I_{\theta}(X) = \frac{1}{1-\theta} \log\left(\sum_{m=0}^{\infty} b_m \int_{-\infty}^{\infty} h(x)^{\theta} H(x)^m \, dx\right), \qquad \mathbf{(15)}$$

where

$$b_m = [\beta(\delta,\eta)]^{-\theta} \sum_{j,k=0}^{\infty} \binom{k+\theta\delta-\theta}{j} \binom{\theta(\eta-1)}{k} \frac{(-1)^{j+k}[log(\alpha)]^{m+\theta}(k+\delta\theta-j)^m}{m!(\alpha-1)^{k+\delta\theta}}.$$

4.5 Quantile Function

To find the quantile function (QF) for this family of distributions, using the inverse of the function $H(x)$ express x in terms of p to get the QF :

$$Q(p) = H^{-1}\left(\frac{\log(1+I_y^{-1}(p;\delta,\eta)(\alpha-1))}{\log(\alpha)}\right),$$

where $I_y^{-1}(p;\delta,\eta)$ is the inverse of the regularized incomplete beta function.

5 Maximum Likelihood Estimation

Let $x_1, \dots, x_n$ be the observed random sample from the BAPEx distribution with parameters α, η, λ and δ.

Let $\boldsymbol{\theta} = (\alpha, \eta, \lambda, \delta^T)^T$ be the $(p \times 1)$ parameter vector. Then, the log-likelihood function for $\boldsymbol{\theta}$, say $\ell = \ell(\boldsymbol{\theta})$, is given by

$$\ell = -n\log\left[\beta(\delta,\eta)\right] + n\log\left[\log(\alpha)\right] - n\log(\alpha-1)$$
$$+\sum_{i=1}^{n}\left[\log\left(\lambda\, e^{-\lambda\, x_i}\right) + z_i log(\alpha) + (\delta-1)\log\left(\frac{\alpha^{z_i}-1}{\alpha-1}\right)\right.$$
$$\left. + (\eta-1)\log\left(1-\frac{\alpha^{z_i}-1}{\alpha-1}\right)\right].$$

Th score vector components, say $\mathrm{U}(\theta) = (\frac{\partial\ell}{\partial\theta} = \frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\delta}, \frac{\partial\ell}{\partial\eta}\frac{\partial\ell}{\partial\lambda})^{\mathrm{T}} = (\mathrm{U}_\alpha, \mathrm{U}_\delta, \mathrm{U}_\eta, \mathrm{U}_\lambda)^{\mathrm{T}}$,

are given by

$$U_\delta = \frac{-na_1}{\beta(\delta,\eta)} + \sum_{i=1}^{n}\log\left(\frac{\alpha^{z_i-1}}{\alpha-1}\right),$$

$$U_\eta = \frac{-na_2}{\beta(\delta,\eta)} + \sum_{i=1}^{n}\log\left(1-\frac{\alpha^{z_i-1}}{\alpha-1}\right),$$

$$U_\alpha = \frac{n}{\alpha\log(\alpha)} - \frac{n}{\alpha-1} + \sum_{i=1}^{n}\left[\frac{z_i}{\alpha} + (\delta-1)\left(\frac{z_i\,\alpha^{z_i-1}}{\alpha^{z_i}-1} - \frac{1}{\alpha-1}\right) - (\eta-1)\frac{\alpha^{z_i}\log(\alpha)x_i e^{-\lambda x_i}}{(\alpha-1)\left(1-\frac{\alpha^{z_i-1}}{\alpha-1}\right)}\right],$$

$$U_\lambda = \sum_{i=1}^{n}\left[\frac{1}{\lambda} - x_i + x_i e^{-\lambda x_i}\log(\alpha) + (\delta-1)\frac{\alpha^{z_i}\log(\alpha)x_i e^{-\lambda x_i}}{\alpha^{z_i}-1} - (\eta-1)\frac{\alpha^{z_i}\log(\alpha)x_i e^{-\lambda x_i}}{(\alpha-1)\left(1-\frac{\alpha^{z_i}-1}{\alpha-1}\right)}\right],$$

where

$a_1 = \frac{\partial\beta(\delta,\eta)}{\partial\delta}$, $a_2 = \frac{\partial\beta(\delta,\eta)}{\partial\eta}$ and $z_i = 1 - e^{-\lambda x_i}$. Setting the nonlinear system of equations $U_\alpha = U_\delta = U_\lambda = U_\eta = 0$ and solving them simultaneously yields the

MLE $\hat{\theta} = (\hat{\alpha}, \hat{\delta}, \hat{\lambda}, \hat{\eta}^T)^T$ of the $\theta = (\alpha, \delta, \lambda, \eta^T)^T$. The model parameters are estimated using the MLE method by solving the first-order derivative equations of the log-likelihood function. Once the estimates are obtained, the Fisher information matrix is computed from the second derivatives (the negative Hessian) of the log-likelihood function. The inverse of this matrix represents the variance–covariance matrix of the estimators, from which the standard errors (SEs) are derived as the square roots of its diagonal elements.

6. Simulation Study

Now, we provided detailed simulation results to explore the performances of the ML estimation in estimating the parameters of the BAPEx model. We considered several sample sizes and different values of the parameters, that is, $n = \{50, 80, 200, 400\}$. The behavior of the different estimates is compared with respect to their: average absolute bias ($|BIAS|$), $|BIAS| = \frac{1}{N}\sum_{i=1}^{N}|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}|$, mean square errors (MSE), $MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^2$, and mean relative errors (MRE), MRE $= \frac{1}{N}\sum_{i=1}^{N}|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}|/\boldsymbol{\theta}$. Table 2 show the simulation results, average ML estimates of the parameters, $|BIAS|$, MSE, and MRE, of the BAPEx parameters using different approaches. These results showed that estimates are very close to their true values and have small biases, MSE and MRE. The results illustrated that the biases, MSE, and MRE decrease as n increases, showing that the introduced estimators are consistent. The results were obtained through statistical analyses performed using the R software environment.

Table 2: The AEs, MSE, BIAS and MRE of the BAPEx parameters for different values of the parameters and n.

n	$\alpha = 0.4, \delta = 1.5, \eta = 1.75, \lambda = 1.75$				
50	AEs	27.45821	2.07983	12.20087	2.78534
80		1.47232	1.77045	2.13959	2.44588
200		0.58832	1.56584	1.73166	2.01634
400		0.48298	1.51885	1.73609	1.83471
50	MSE	11.611	4.04283	91.14586	10.16723
80		7.7943	1.35982	19.72764	5.81346
200		0.75376	0.12031	0.50563	1.69564
400		0.20903	0.03061	0.10045	0.47244
50	BAIS	27.32195	0.83765	11.15317	1.36319
80		1.31913	0.49489	0.97941	0.94933
200		0.39946	0.21178	0.36307	0.47609
400		0.24517	0.13021	0.21587	0.26146
50	MRE	68.30488	0.55844	6.37324	0.77897
80		3.29784	0.32992	0.55966	0.54247
200		0.99864	0.14119	0.20747	0.27205
400		0.61293	0.08681	0.12336	0.14941
n	$\alpha = 1.5, \delta = 0.5, \eta = 0.75, \lambda = 0.75$				
50	AEs	2.24801	0.53101	0.74150	0.94102
80		1.94406	0.51455	0.74588	0.86309
200		1.60473	0.50645	0.75122	0.79366
400		1.51515	0.50389	0.75549	0.76296
50	MSE	15.05972	0.02085	0.10078	0.44496
80		12.67606	0.00906	0.06711	0.21826
200		0.7864	0.00326	0.02421	0.07333
400		0.25966	0.00146	0.01255	0.02638
50	BAIS	1.34964	0.09877	0.19468	0.36022
80		0.94719	0.07017	0.1613	0.27106
200		0.46219	0.04372	0.10892	0.16314
400		0.28944	0.02974	0.07755	0.10533

50	MRE	0.89976	0.19754	0.25957	0.48029
80		0.63146	0.14033	0.21507	0.36141
200		0.30813	0.08744	0.14523	0.21752
400		0.19296	0.05948	0.1034	0.14044
n	$\alpha = 0.25, \delta = 0.6, \eta = 0.6, \lambda = 0.25$				
50	AEs	0.35154	0.66482	0.46038	0.44147
80		0.27717	0.64664	0.49155	0.37917
200		0.23563	0.61926	0.54155	0.31523
400		0.22998	0.60912	0.56417	0.29873
50	MSE	0.2982	0.1369	0.19342	0.22966
80		0.21585	0.10004	0.16634	0.17268
200		0.13157	0.05782	0.12188	0.10497
400		0.09287	0.04544	0.09634	0.07884
50	BAIS	0.27427	0.0438	0.05584	0.12292
80		0.10271	0.02078	0.04366	0.06867
200		0.02807	0.0063	0.02597	0.02723
400		0.0134	0.00621	0.01778	0.02058
50	MRE	1.19279	0.22816	0.32237	0.91865
80		0.86341	0.16674	0.27723	0.69071
200		0.52629	0.09636	0.20314	0.4199
400		0.37149	0.07573	0.16057	0.31537

7 An Application to Real Data

In this section, The BAPEx model is fitted to real data and compared with other existing distributions. In order to compare the fits of the distributions, we consider various measures of goodness-of-fit including the Akaike information criterion (AIC) (Akaike, 1973), Kolmogorov -Smirnov (K-S) statistics (with its $p-$value) and $\hat{\ell}$, where $\hat{\ell}$, is the maximized log-likelihood. The dataset comprises of 74 observations, specifically referring to gauge lengths of 20 mm. The analysis of this data studied by Kundu and Raqab (2009). Data set : The gauge lengths of 20 mm data: 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585. Here, we use this data to compare the BAPEx model with other models, namely: generalized exponential (GEx) (Gupta and Kundun, 2001), Kumaraswamy transmuted-exponential (Kw-TEx) (Afify et al., 2016), generalized transmuted exponential (GTEx) (Nofal et al., 2017), generalized transmuted generalized exponential (GTGEx) (Nofal et al., 2017), alpha power exponentiated exponential (APExEx) (Afify et al., 2020), the exponentiated generalized alpha power exponential (EGAPEx) (ElSherpieny and Almetwally, 2022), Alpha power exponentiated inverse exponential (APEIEx) (Kargboet al., 2023) and exponentiated generalized Weibull exponential distribution (EGWEx) (Abonongo and Abonongo, 2024), distribution. We compare the fits of the BAPEx model with the EGAPEx, KWT-Ex, APExEx, APEIEx, GEx, GT-GEx, EGWEx and GT-GGEx models. Table 3 list the numerical values of AIC, K-S (P-value) and $\hat{\ell}$ for the models fitted to the gauge lengths of 20 mm. The figures in this table indicate that the BAPEx model has the lowest values for goodness-of-fit statistics for the gauge lengths of 20 mm among the fitted models. So, the BAPEx model could be chosen as the best model. The MLEs and their corresponding standard errors of the model parameter are given in Table 4. The results were obtained through statistical analyses performed using the R software environment.

Table 3: Findings from the fitted distributions to the gauge lengths of 20 mm dataset.

Model	AIC	$\hat{\ell}$	K-S(P-value)
BAPEx	110.3749	51.18743	0.0560 (0.9743)
EGAPEx	110.7216	51.36079	0.0571 (0.9722)
Kw-TEx	111.0739	51.5369	0.0583 (0.9720)
GT-GEx	113.2988	52.64939	0.0687 (0.8765)
EGWEx	114.2961	52.14805	0.0578 (0.9657)
GT-GGEx	115.2988	52.6493	0.0686 (0.8766)
APExEx	115.3082	54.65412	0.0697 (0.8648)
GEx	121.6065	58.80327	0.0953 (0.5120)
APEIEx	297.6644	145.8322	0.4695(0.0030)

Table 4: MLEs and the corresponding SEs of the gauge lengths of 20 mm data.

Model	parameter (SEs)					
BAPEx	α	δ	η	λ		
	0.211 (0.631)	2568.307 (3925.267)	4.306 (8.924)	429.294 (4318.758)		
EGAPEx	s	P	B	λ		
	5256.025 (170.3)	8865 (1819.498)	5.716 (1.145)	0.08997 (0.00649)		
GT-GEx	λ	α	B	a		
	2.819 (0.304)	108.731 (73.534)	754.477 (612.794)	0.8419 (0.093)		
APEIEx	α	a	B			
	3236.676 (3786.408)	1.265 (24.369)	0.231 (4.459)			
GT-GGEx	λ	a	B	α	S	
	10.394 (482.8)	0.46 (485.8)	72.577 (3371)	0.841 (0.09313)	2.819 (0.3041)	
Kw-TEx	a	b	λ	α		
	5.896 (0.519)	78.44 (206.267)	19.273 (7.366)	0.009 (0.0517)		
EGWEx	α	d	B	b	c	a
	0.381 (10.1068)	4.499 (1.351)	1.9123 (108.873)	1.596 (0.907)	6.607 (742.411)	1.735 (81.823)
APExEx	α	a	C			
	46.324 (65.8188)	2.521 (0.2173)	91.046 (52.846)			
GEx	α	S				

	89.431 (32.472)	2.019 (0.172)

Figure 3 illustrates the goodness-of-fit of the BAPEx distribution to the observed data. The PDF, CDF and SF plots show a close agreement between the empirical and theoretical curves. Moreover, the P–P plot demonstrates that the observed and expected values are nearly aligned along the 45-degree line, confirming the adequacy of the BAPEx model in describing the data.

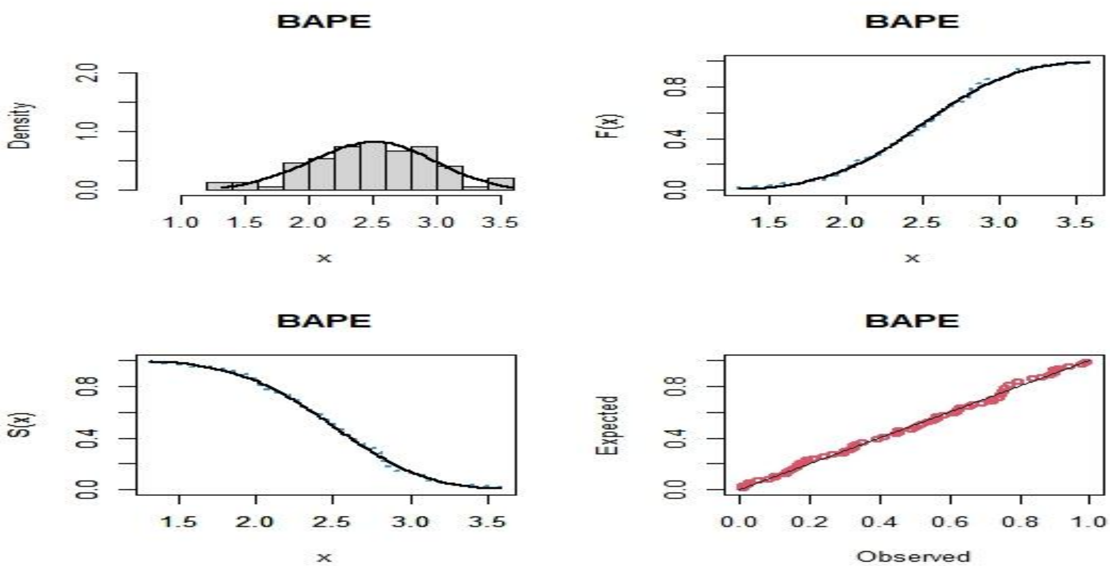

Figure 3: The fitted BAPEx PDF, CDF, SF, and P–P plots for gauge lengths dataset.

8 Conclusions

In this paper, we introduce a new four-parameter distribution called the beta alpha power exponential (BAPEx) distribution. The statistical properties of the new model are derived. Further, the BAPEx distribution parameters are estimated by the maximum likelihood method. A simulation study is conducted to explore the performance of the maximum likelihood method. Finally, the practical importance of the BAPEx distribution is studied by analyzing a real-life dataset. Goodness-of-fit statistics for the analyzed data set showed that our BAPEx distribution provides a better fit in comparison with other rival distributions.

References

1. Abonongo, A. I. L. and Abonongo, J. (2024). Exponentiated generalized Weibull exponential distribution: properties, estimation and applications. Computational Journal of Mathematical and Statistical Sciences, 3(1), 57-84.
2. Afify, A. Z., Gemeay, A. M. and Ibrahim, N. A. (2020). The heavy-tailed exponential distribution: risk measures, estimation, and application to actuarial data. Mathematics, 8(8), 1276.
3. Afify, A.Z., Yousof, H. and Nadarajah, S. (2017). The beta transmuted-H family for lifetime data. Statistics and its Interface, 10(3), 505-520.
4. Afify, A. Z., Cordeiro, G. M., Yousof, H. M. and Nofal, Z. M. and Alzaatreh, A. (2016). The Kumaraswamy transmuted-G family of distributions: properties and applications. Journal of Data Science, 14(2), 245-270.
5. Akaike, H. (1973). Information theory and an extension of the maximum likelihood principle. In International Symposium on Information Theory, 2 nd, Tsahkadsor, Ar-menian SSR, 267.281.
6. Alizadeh, M., Cordeiro, G. M., Brito, E. D. and B. Demétrio, C. G. (2015). The beta Marshall-Olkin family of distributions. Journal of Statistical distributions and Applications, 2, 1-18.
7. Anderson, T. W. and Darling, D. A. (1952). Asymptotic theory of certain" good-ness of fit" criteria based on stochastic processes. The annals of mathematical statistics, 193-212.
8. Awodutire, P. O., Balogun, O. S., Olapade, A. K. and Nduka, E. C. (2021). The modified beta transmuted family of distributions with applications using the exponential distribution. Plos one, 16(11), e0258512.
9. Cordeiro, G. M. and de Castro, M. (2011). A new family of generalized distributions. Journal of statistical computation and simulation, 81(7), 883-898.

10. Cordeiro, G. M., Ortega, E. M. and da Cunha, D. C. (2013). The exponentiated generalized class of distributions. Journal of data science, 11(1), 1-27.
11. Dias, C. R., Alizadeh, M. and Cordeiro, G. M. (2018). The beta Nadarajah-Haghighi distribution. Hacettepe Journal of Mathematics and Statistics, 47(5), 1302-1320.
12. ElSherpieny, E. A. and Almetwally, E. M. (2022). The exponentiated generalized alpha power family of distribution: Properties and applications. Pakistan Journal of Statistics and Operation Research, 8(2) 349-367.
13. Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. Communications in Statistics-Theory and methods, 31(4), 497-512.
14. Gradshteyn, I. S. and Ryzhik, I. M. (2014). Table of integrals, series, and products. Academic press.
15. Gupta, R. C., Gupta,P. L. and Gupta, R. D. (1998). Modeling failure time data by Lehman alternatives. Communications in Statistics-Theory and methods, 27(4), 887-904.
16. Gupta, R. D. and Kundu, D. (2001). Generalized exponential distribution: different method of estimations. Journal of Statistical Computation and Simulation, 69(4), 315-337.
17. Hogg, R. V., McKean, J. & Craig, A. T. (2005). Introduction to Mathematical Statistics. Pearson Education.
18. Kargbo, M., Waititu, A. G. and Wanjoya, A. K. (2023). The alpha power exponentiated inverse exponential distribution and its application on Italy’s Covid-19 mortality rate data. Int J Sci Res Eng Develop, 6(4), 130-139.
19. Kundu, D. and Raqab, M. Z. (2009). Estimation of R= P (Y< X) for Three parameter Weibull distribution. Statistics Probability Letters, 79(17), 1839-1846.

20. Mahdavi, A. and Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. Communications in Statistics-Theory and Methods, 46(13), .6543-6557.
21. Nofal, Z. M., Afify, A. Z., Yousof, H. M. and Cordeiro, G. M. (2017). The generalized transmuted-G family of distributions. Communications in Statistics-Theory and Methods, 46(8), 4119-4136.
22. Paranaíba, P. F., Ortega, E. M., Cordeiro, G. M. and Pescim, R. R. (2011). The beta Burr XII distribution with application to lifetime data. Computational statistics data analysis, 55(2), 1118-1136.
23. Pescim, R. R., Demétrio, C. G., Cordeiro, G. M., Ortega, E. M. and Urbano, M. R. (2010). The beta generalized half-normal distribution. Computational statistics data analysis, 54(4), 945-957.
24. Semary, H. E., Hussain, Z., Hamdi, W. A., Aldahlan, M. A., Elbatal, I. and Nagarjuna, V. B. (2024). Alpha–beta-power family of distributions with applications to exponential distribution. Alexandria Engineering Journal, 100, 15-31.
25. Sadiq, I. A., Doguwa, S. I., Yahaya, A. and Garba, J. (2023). New generalized odd Frechet-G (NGOF-G) family of distribution with Statistical properties and applications. UMYU Scientifica, 2(3), 100-107.
26. Yousof, H. M., Rasekhi, M., Afify, A. Z., Ghosh, I., Alizadeh, M and Hamedani, G. G. (2017). The beta Weibull-G family of distribution: theory characterizations and application. Pakistan Journal of Statistics, 33(2), 95-116.
27. Zografos, K. (2008). On some beta generated distributions and their maximum entropy characterization: The beta-Weibull distribution. Advances in Inequalities from Probability Theory Statistics, 237-260.