Estimation of Generalized Inverted Exponential Distribution under Generalized Hybrid Censoring Scheme

by

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Abstract

In this paper, we consider the estimation of the parameters of the generalized inverted exponential distribution when data are generalized hybrid type-I censored samples. The maximum likelihood estimators of the parameters and the confidence interval have been obtained. Additionally, the parameters have been estimated using the Bayesian method with the squared error and linear-exponential loss functions , considering a gamma prior and the corresponding posterior distributions, Bayes estimators of the unknown parameters cannot be calculated in closed forms. The Markov Chain Monte Carlo method, namely the Metropolis-Hastings algorithm, has been used to derive approximations for the simulation study.We achieve the highest posterior density (HPD) credible intervals.The proposed estimators in the maximum likelihood and Bayesian methods have been compared. Finally, a real data set has been analyzed for illustrative purposes.

Keywords Generalized inverted exponential distribution,Generalized hybrid type-I censored samples,Maximum likelihood estimation,Bayes estimation,Markov Chain Monte Carlo.

1 Introduction

There is no doubt that estimations based on complete samples are more accurate. However, it is inevitable to use censoring for lifetime experiments due to time constraints and expense reduction. type-I and type-II censoring are usually considered as two fundamental methods to conduct lifetime experiments, Where we terminate these experiments at a certain time point or upon the occurrence of a certain number of failures. With the rapid development of science and technology, products have higher reliability and longer life spans, resulting in a longer time of life-testing to obtain sufficient failure samples.

In order to cut down the life-testing duration, Epstein (1954) carried out a hybrid type-I censoring scheme that could be considered as a combination of those two fundamental censoring schemes .Under this scheme, lifetime experiments operate after a specific point of time and the number of failures is pre-fixed. As long as either of these occurs, the test will be terminated. However, this scheme also has limitations as it has a possibility that extremely few failures occur before the pre-determined time. As a result, it may be impractical to make statistical inferences under such a scheme.

In order to overcome this disadvantage and improve the efficiency of estimators in the lifetesting experiment as well as to guarantee that a certain number of failures appear before the end of the experiment as well as saving the time of testing and the cost resulted from failures of units,Chandrasekar et al .(2004) introduced a generalized hybrid type-I censoring scheme (GHCS-I).Generalized hybrid type-I censoring assures a minimum number of failures, Which could mitigate the short back that exists in hybrid type-I censoring . Some authors have studied the estimation parameters of some distribution under GHCS-I, such as Ahmad (2019), Rabie and Li(2019), Zhang et al. (2021),Dhamecha et al. (2021), Mahmoud et al. (2021),and Liu and Zhang (2021).

The GHCS-I described as follows : Fix integers $k,m \in (1,2,...,n)$ such that $k < m < n$,and time $T \in (0,\infty)$.The termination time of the experiment is $T^* = min\{x_{(m)},max\{x_{(k)},T\}\}$.If the $k - th$ failure occurs before time T , terminate the experiment at $min\{x_{(m)},T\}$. If the $k - th$ failure occurs after time T . terminate the experiment at $x_{(k)}$.

Under the GHCS-I, the observed data will be one of the following cases of observations:-

Case-I:$\{x_{(1)} < \cdots < x_{(k)}\}$, if $T < x_{(k)} < x_{(m)}$

Case-II:$\{x(1) < \cdots < x(k) < x(d) < \cdots < x(m)\}$, if $x(k) < T < x(m)$

Case-III:$\{x(1) < \cdots < x(k) < x(m)\}$, if $x(k) < x(m) < T$.

The likelihood function can be rewritten as follows;

$$L = \begin{cases} \frac{n!}{(n-k)!} \prod_{i=1}^{k} f(x_{(i)}))[1 - F(x_{(k)})]^{n-k}, & d = 0.1..., k-1 \\ \frac{n!}{(n-d)!} \prod_{i=1}^{d} f(x_{(i)}))[1 - F(T)]^{n-d}, & d = k, k+1, ..., m-1 \\ \frac{n!}{(n-m)!} \prod_{i=1}^{m} f(x_{(i)}))[1 - F(x_{(m)})]^{n-m}, & d = m \end{cases} \quad \textbf{(1)}$$

The generalized inverted exponential distribution (GIED) was introduced first by Abouammoh and Alshingiti (2009). It was a generalized form of the inverted exponential distribution .The GIED has good statistical and reliability properties. It fits various shapes of failure rates . Dey et al. (2014),the GIED is widely applied in research related life testing, horse racing, supermarket queues, sea currents, wind speeds, and many more Kotz and Nadarajah (2000). the probability density function (pdf) , cumulative distribution function(cdf) ,respectively, as follows;

$$f(x) = \left(\frac{\alpha\theta}{x^2}\right) \exp\left(\frac{-\theta}{x}\right) \left[1 - \exp\left(\frac{-\theta}{x}\right)\right]^{\alpha-1}, \quad x > 0, \quad \alpha, \theta > 0 \qquad \textbf{(2)}$$

$$F(x) = 1 - \left[1 - \exp\left(\frac{-\theta}{x}\right)\right]^{\alpha}, \qquad x > 0, \quad \alpha, \theta >$$

$$H(x) = \left(\frac{\alpha\theta}{x^2}\right) \left(e^{\frac{\theta}{x}} - 1\right)^{-1} \qquad 0 \qquad \textbf{(3)}$$

The α is shape parameter and θ is scale parameter. Figure (1) the PDF of GIED and figure (2) the hazard rate function of GIED.

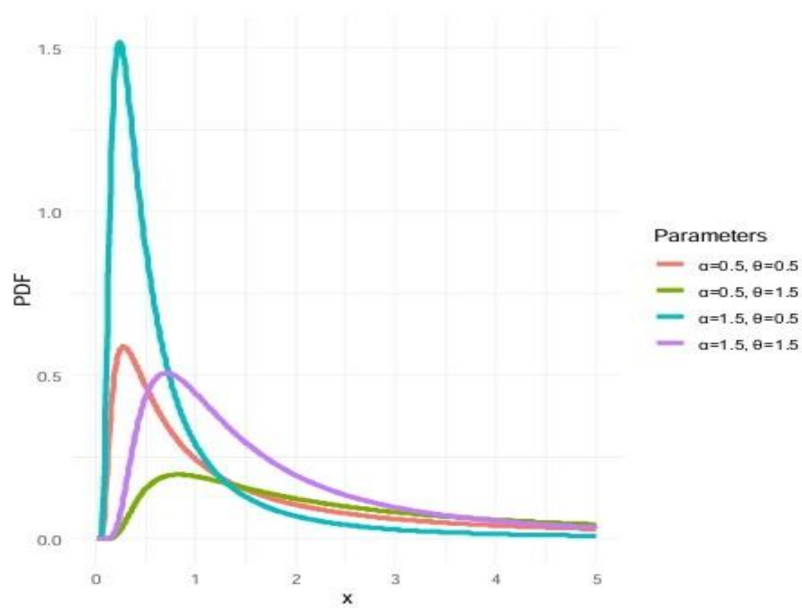

Figure 1: PDF of GIED

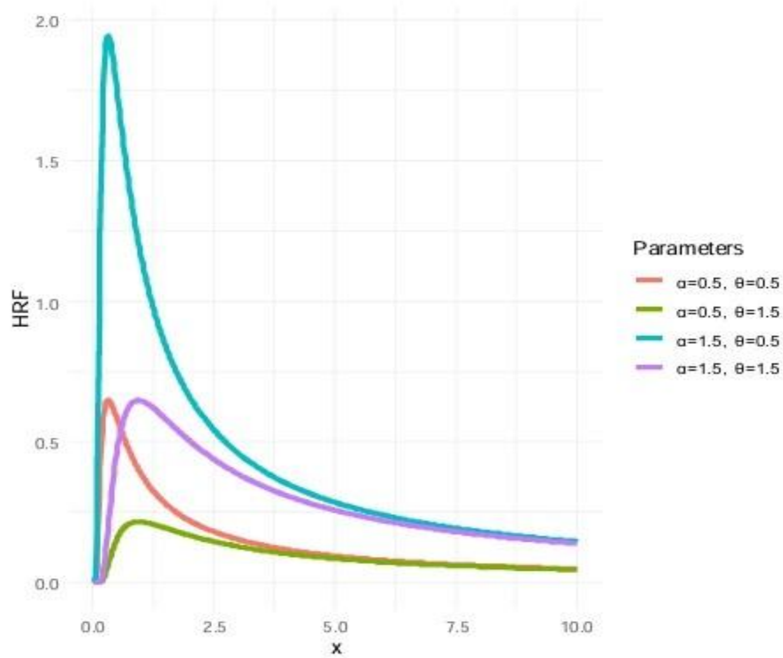

Figure 2: hazart rate function of GIED

This paper is organized as follows: In Section **2**, the maximum likelihood estimates for unknown parameters under the GHCS-I are derived. In Section **3**,asymptotic confidence interval. In Section **4**, Bayes estimates of the unknown parameters under squared error (SE) and linearexponential (LINEX) loss function by using Markov Chain Monte Carlo (MCMC) method . In Section **5**, a simulation study is implemented . In Section **6** ,the analysis of real date set is presented . In Section **7**, concluding remarks are discussed .

2 Maximum likelihood estimation

In this section we drive the maximum likelihood estimator (MLE) of the unknown parameters of GIED (α,θ) under GHCS-I, the likelihood function for three cases by equation (1)

Based on the pdf and the cdf of GHCS-I using GIED by equations (2) and (3), respectively, then the likelihood function can be rewritten as follows

$$L \propto \left(\alpha^k \theta^k\right) \exp\left(-\theta \sum_{i=1}^{k} \left(\frac{1}{x_i}\right)\right) \prod_{i=1}^{k} \left(\frac{1}{x_i^2}\right) \left[1-\exp\left(-\frac{\theta}{x_i}\right)\right]^{\alpha-1} \left[1-\exp\left(-\frac{\theta}{x_k}\right)\right]^{\alpha(n-k)}, \quad (4)$$

The logarithm of equation (4) can be written as:

$$\begin{aligned} LnL \propto & k \ln \alpha + k \ln \theta - \theta \sum_{i=1}^{k} \left(\frac{1}{x_i}\right) + \sum_{i=1}^{k} \ln \left(\frac{1}{x_i^2}\right) + (\alpha - 1) \sum_{i=1}^{k} \ln \left[1-\exp\left(-\frac{\theta}{x_i}\right)\right] \\ & + \alpha(n-k) \ln \left[1-\exp\left(-\frac{\theta}{x_k}\right)\right], \end{aligned} \quad (5)$$

Taking derivatives with respect to α and θ of equation (5), and equality to zero, we obtain the following

$$\frac{\partial \operatorname{In} L}{\partial \alpha} = \left(\frac{k}{\hat{\alpha}}\right) + \sum_{i=1}^{k} \ln \left[1-\exp\left(-\frac{\hat{\theta}}{x_i}\right)\right] + (n-k) \ln \left[1-\exp\left(-\frac{\hat{\theta}}{x_k}\right)\right] = 0 \quad \textbf{(6)}$$

$$\frac{\partial \operatorname{In} L}{\partial \theta} = \left(\frac{k}{\hat{\theta}}\right) - \sum_{i=1}^{k} \left(\frac{1}{x_i}\right) + (\hat{\alpha}-1) \sum_{i=1}^{k} \frac{(1/x_i)\exp\left(-\hat{\theta}/x_i\right)}{\left[1-\exp\left(-\hat{\theta}/x_i\right)\right]} + \hat{\alpha}(n-k) \frac{(1/x_k)\exp\left(-\hat{\theta}/x_k\right)}{\left[1-\exp\left(-\hat{\theta}/x_k\right)\right]} = 0 \quad (7)$$

$$\hat{\alpha} = \frac{-k}{\sum_{i=1}^{k} \ln \left[1-\exp\left(-\frac{\hat{\theta}}{x_i}\right)\right] + (n-k) \ln \left[1-\exp\left(-\frac{\hat{\theta}}{x_k}\right)\right]} \quad \textbf{(8)}$$

Similarly, for case II and III in a GHCS-I, the estimate of α and θ can be written as:

$$\hat{\alpha} = \begin{cases} \frac{-d}{\sum_{i=1}^{d} \ln\left[1-\exp\left(-\frac{\theta}{x_i}\right)\right]+(n-d)\ln\left[1-\exp\left(-\frac{\theta}{T}\right)\right]}, & caseII \\ \frac{-m}{\sum_{i=1}^{m} \ln\left[1-\exp\left(-\frac{\theta}{x_i}\right)\right]+(n-m)\ln\left[1-\exp\left(-\frac{\theta}{x_m}\right)\right]} \end{cases}$$

(9) ,

caseIII.

and

$$\hat{\theta} = \begin{cases} \frac{d}{\hat{\theta}} - \sum_{i=1}^{d} \frac{1}{x_i} + (\hat{\alpha}-1)\sum_{i=1}^{d} \frac{(1/x_i)\exp\left(-\hat{\theta}/x_i\right)}{1-\exp\left(-\hat{\theta}/x_i\right)} + \hat{\alpha}(n-d)\frac{(1/T)\exp\left(-\hat{\theta}/T\right)}{1-\exp\left(-\hat{\theta}/T\right)}, caseII \\ \frac{m}{\hat{\theta}} - \sum_{i=1}^{m} \frac{1}{x_i} + (\hat{\alpha}-1)\sum_{i=1}^{m} \frac{(1/x_i)\exp\left(-\hat{\theta}/x_i\right)}{1-\exp\left(-\hat{\theta}/x_i\right)} + \hat{\alpha}(n-m)\frac{(1/x_m)\exp\left(-\hat{\theta}/x_m\right)}{1-\exp\left(-\hat{\theta}/x_m\right)}, caseIII. \end{cases}$$

(10)

Equation 10 is very hard to evaluate theoretically and a numerical procedure is needed to solve this equation numerically.

3 Asymptotic Confidence Interval

The asymptotic variance-covariance matrix of $(\hat{\alpha}, \hat{\theta})$ is obtained by inverting the information matrix with elements that are negatives of expected values of the second order derivatives of logarithms of the likelihood function.

$$I_{ij}(\lambda) = E\left[\frac{-\partial^2 \ln L}{\partial\lambda_i \partial\lambda_j}\right], \qquad i,j = 1,2,\ldots,v$$

$$I_0^{-1} = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial\alpha^2} & -\frac{\partial^2 \ln L}{\partial\alpha\partial\theta} \\ -\frac{\partial^2 \ln L}{\partial\alpha\partial\theta} & -\frac{\partial^2 \ln L}{\partial\theta^2} \end{bmatrix}^{-1}_{\alpha=\hat{\alpha},\theta=\hat{\theta}}$$

$$I_0^{-1} = \begin{bmatrix} V(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\theta}) \\ Cov(\hat{\alpha},\hat{\theta}) & V(\hat{\theta}) \end{bmatrix}$$

The element of Fisher information matrix are given as follows:

CaseI

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \frac{-k}{\hat{\alpha}^2},$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \theta} = \sum_{i=1}^{k} x_i^{-1} e^{-\hat{\theta}/x_i} [1 - e^{-\hat{\theta}/x_i}]^{-1} + (n-k) x_k^{-1} e^{-\hat{\theta}/x_k} [1 - e^{-\hat{\theta}/x_k}]^{-1}$$

,

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-k}{\hat{\theta}^2} - (\hat{\alpha} - 1) \sum_{i=1}^{k} x_i^{-2} e^{-2\hat{\theta}/x_i} [1 - e^{-\hat{\theta}/x_i}]^{-2} - \alpha (n-k) x_k^{-2} e^{-2\hat{\theta}/x_k} [1 - e^{-\hat{\theta}/x_k}]^{-2}$$

CaseII

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \frac{-d}{\hat{\alpha}^2},$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \theta} = \sum_{i=1}^{d} x_i^{-1} e^{-\hat{\theta}/x_i} [1 - e^{-\hat{\theta}/x_i}]^{-1} + (n-d) T^{-1} e^{-\hat{\theta}/T} [1 - e^{-\hat{\theta}/T}]^{-1}$$

,

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-d}{\hat{\theta}^2} - (\hat{\alpha} - 1) \sum_{i=1}^{d} x_i^{-2} e^{-2\hat{\theta}/x_i} [1 - e^{-\hat{\theta}/x_i}]^{-2} - \alpha (n-d) T^{-2} e^{-2\hat{\theta}/T} [1 - e^{-\hat{\theta}/T}]^{-2}$$

.

CaseIII

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \frac{-m}{\hat{\alpha}^2},$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \theta} = \sum_{i=1}^{m} x_i^{-1} e^{-\hat{\theta}/x_i}[1-e^{-\hat{\theta}/x_i}]^{-1} + (n-m)x_m^{-1}e^{-\hat{\theta}/x_m}[1-e^{-\hat{\theta}/x_m}]^{-1},$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-m}{\hat{\theta}^2} - (\hat{\alpha}-1)\sum_{i=1}^{m} x_i^{-2} e^{-2\hat{\theta}/x_i}[1-e^{-\hat{\theta}/x_i}]^{-2} - \alpha(n-m)x_m^{-2}e^{-2\hat{\theta}/x_m}[1-e^{-\hat{\theta}/x_m}]^{-2}.$$

Thus,two-sided $100(1-\gamma)\%$ confidence interval of α and θ respectively,

$$\hat{\alpha} \pm z_{\frac{\gamma}{2}}\sqrt{V(\hat{\alpha})}, \qquad \hat{\theta} \pm z_{\frac{\gamma}{2}}\sqrt{V(\hat{\theta})}$$

Where γ is a significance level.

4 Bayesian estimation

In this section, assume that α and θ are independent and follow a gamma prior distribution:

$$\pi(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)}\alpha^{a_1-1}e^{-b_1\alpha}, a_1, b_1 > 0$$

$$\pi(\theta) = \frac{b_2^{a_2}}{\Gamma(a_2)}\theta^{a_2-1}e^{-b_2\theta}, a_2, b_2 > 0$$

Thus, the joint prior distribution is obtained as

$$\pi(\alpha,\theta) = \frac{b_1^{a_1}}{\Gamma(a_1)}\frac{b_2^{a_2}}{\Gamma(a_2)}\alpha^{a_1-1}\theta^{a_2-1}e^{-(b_1\alpha+b_2\theta)}, a_1, b_1, a_2, b_2 > 0. \quad \textbf{(11)}$$

and the likelihood function is shown as

$$L = \begin{cases} \frac{n!}{(n-k)!}\alpha^k\theta^k e^{(\frac{-\theta}{x_i})} \prod_i^k (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_k^{\alpha(n-k)}, & d < k \\ \frac{n!}{(n-d)!}\alpha^d\theta^d e^{(\frac{-\theta}{x_i})} \prod_i^d (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_T^{\alpha(n-d)}, & k \le d < m \\ \frac{n!}{(n-m)!}\alpha^m\theta^m e^{(\frac{-\theta}{x_i})} \prod_i^m (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_m^{\alpha(n-m)}, & d \ge m \end{cases} \quad \textbf{(12)}$$

$$W_i = 1 - e^{(\frac{-\theta}{x_i})}, W_k = 1 - e^{(\frac{-\theta}{x_k})}, W_T = 1 - e^{(\frac{-\theta}{T})}, W_m = 1 - e^{(\frac{-\theta}{x_m})},$$

where

On the basis of Bayesian method, multiply equation (11) by equation (12) to obtain the joint posterior distribution

$$\pi(\alpha,\theta|x) = \begin{cases} K^{-1}\alpha^{k+a_1-1}\theta^{k+a_2-1}e^{-[(b_1\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^k (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_k^{\alpha(n-k)}, & d < k \\ K^{-1}\alpha^{d+a_1-1}\theta^{d+a_2-1}e^{-[(b_1\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^d (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_T^{\alpha(n-d)}, & k \le d < m \\ K^{-1}\alpha^{m+a_1-1}\theta^{m+a_2-1}e^{-[(b_1\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^m (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_m^{\alpha(n-m)}, & d \ge m \end{cases} \quad (13)$$

where *K* could be written in the following form

$$K = \begin{cases} \int_0^\infty \int_0^\infty \alpha^{k+a_1-1}\theta^{k+a_2-1}e^{-[(b_1\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^k (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_k^{\alpha(n-k)} d\alpha d\theta, & d < k \\ \int_0^\infty \int_0^\infty \alpha^{d+a_1-1}\theta^{d+a_2-1}e^{-[(b_1\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^d (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_T^{\alpha(n-d)} d\alpha d\theta, & k \le d < m \\ \int_0^\infty \int_0^\infty \alpha^{m+a_1-1}\theta^{m+a_2-1}e^{-[(b_1\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^m (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_m^{\alpha(n-m)} d\alpha d\theta, & d \ge m \end{cases}$$

The Bayesian estimator of two unknown parameters under SE loss functions as follows,

$$\hat{\alpha}_{BS} = E(\alpha \mid x) = \int_0^\infty \int_0^\infty \alpha\pi(\alpha,\theta \mid x) d\alpha d\theta$$

$$\hat{\alpha}_{BS} = \begin{cases} K^{-1}\int_0^\infty \int_0^\infty \alpha^{k+a_1}\theta^{k+a_2-1}e^{-[(b_1\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^k (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_k^{\alpha(n-k)} d\alpha d\theta, & d < k \\ K^{-1}\int_0^\infty \int_0^\infty \alpha^{d+a_1}\theta^{d+a_2-1}e^{-[(b_1\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^d (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_T^{\alpha(n-d)} d\alpha d\theta, & k \le d < m \\ K^{-1}\int_0^\infty \int_0^\infty \alpha^{m+a_1}\theta^{m+a_2-1}e^{-[(b_1\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^m (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_m^{\alpha(n-m)} d\alpha d\theta, & d \ge m \end{cases}$$

and

$$\hat{\theta}_{BS} = E(\theta \mid x) = \int_0^\infty \int_0^\infty \theta \pi(\alpha, \theta \mid x) d\alpha d\theta$$

$$\hat{\theta}_{BS} = \begin{cases} K^{-1} \int_0^\infty \int_0^\infty \alpha^{k+a_1-1} \theta^{k+a_2} e^{-[(b_1\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^k (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_k^{\alpha(n-k)} d\alpha d\theta, & d < k \\ K^{-1} \int_0^\infty \int_0^\infty \alpha^{d+a_1-1} \theta^{d+a_2} e^{-[(b_1\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^d (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_T^{\alpha(n-d)} d\alpha d\theta, & k \le d < m \\ K^{-1} \int_0^\infty \int_0^\infty \alpha^{m+a_1-1} \theta^{m+a_2} e^{-[(b_1\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^m (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_m^{\alpha(n-m)} d\alpha d\theta, & d \ge m \end{cases}$$

The Bayesian estimator of two unknown parameters under LINEX loss functions as follows,

$$\hat{\alpha}_{BL} = -\frac{1}{h} \ln E\left(e^{-h\alpha} \mid x\right) = -\frac{1}{h} \ln \int_0^\infty \int_0^\infty e^{-h\alpha} \pi(\alpha, \theta \mid x) d\alpha d\theta$$

$$\hat{\alpha}_{BL} = \begin{cases} K^{-1} \int_0^\infty \int_0^\infty \alpha^{k+a_1-1} \theta^{k+a_2-1} e^{-[((b_1+h)\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^k (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_k^{\alpha(n-k)} d\alpha d\theta, & d < k \\ K^{-1} \int_0^\infty \int_0^\infty \alpha^{d+a_1-1} \theta^{d+a_2-1} e^{-[((b_1+h)\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^d (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_T^{\alpha(n-d)} d\alpha d\theta, & k \le d < m \\ K^{-1} \int_0^\infty \int_0^\infty \alpha^{m+a_1-1} \theta^{m+a_2-1} e^{-[((b_1+h)\alpha+b_2\theta)+(\frac{\theta}{x_i})]} \prod_i^m (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_m^{\alpha(n-m)} d\alpha d\theta, & d \ge m \end{cases}$$

and

$$\hat{\theta}_{BL} = -\frac{1}{h} \ln E\left(e^{-h\theta} \mid x\right) = -\frac{1}{h} \ln \int_0^\infty \int_0^\infty e^{-h\theta} \pi(\alpha, \theta \mid x) d\alpha d\theta$$

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$$\hat{\theta}_{BL} = \begin{cases} K^{-1} \int_0^\infty \int_0^\infty \alpha^{k+a_1-1} \theta^{k+a_2-1} e^{-[(b_1\alpha+(b_2+h)\theta)+(\frac{\theta}{x_i})]} \prod_i^k (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_k^{\alpha(n-k)} d\alpha d\theta, & d < k \\ K^{-1} \int_0^\infty \int_0^\infty \alpha^{d+a_1-1} \theta^{d+a_2-1} e^{-[(b_1\alpha+(b_2+h)\theta)+(\frac{\theta}{x_i})]} \prod_i^d (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_T^{\alpha(n-d)} d\alpha d\theta, & k \le d < m \\ K^{-1} \int_0^\infty \int_0^\infty \alpha^{m+a_1-1} \theta^{m+a_2-1} e^{-[(b_1\alpha+(b_2+h)\theta)+(\frac{\theta}{x_i})]} \prod_i^m (\frac{1}{x_i^2}) W_i^{(\alpha-1)} W_m^{\alpha(n-m)} d\alpha d\theta, & d \ge m \end{cases}$$

Bayes estimator ($\hat{\alpha}_{BS}, \hat{\alpha}_{BL}, \hat{\theta}_{BS}, \hat{\theta}_{BL}$) cannot be expressed in closed form,so we need to employ some approximation method to compute the estimate .We propose to use the MCMC method to obtain the Bayes estimator and highest posterior density (HPD) credible intervals of the unknown parameters . We use Metropolis- Hastings (M-H) algorithm as a general MCMC method with wide applications .The steps of M-H algorithm are carried out as follows:

Algorithm

Step 1: Start with an arbitrary starting point $\lambda^{(0)} = (\alpha^{(0)},\theta^{(0)})$ for which $f\left(\lambda^{(0)} \mid \underline{x}\right) > 0$.Step2: Set J = 1.

Step3: generate $\lambda^{(*)}$ with proposal distribution

(a) Evaluate the acceptance probabilities by

$$\rho\left(\lambda^{(J-1)}, \lambda^{*}\right) = \min\left[1, \frac{f\left(\lambda^{(*)}|\underline{x}\right)}{f\left(\lambda^{(J-1)}|\underline{x}\right)}\right].$$

(b) Generate a sample from uniform distribution, i.e., $U \sim U(0,1)$.

(c) If $U \leq \rho\left(\lambda^{(J-1)}, \lambda^{*}\right)$, accept proposal and set $\lambda^{(J)} = \lambda^{*}$, else set $\lambda^{(J)} = \lambda^{(J-1)}$. Step 4: Set J = J + 1

Step 5: Repeat steps 2-4 for M times until to get M samples. Step 6: Bayesian estimates of the parameter λ under SE loss function ,

$$\hat{\lambda}_{BS} = \frac{1}{M-M^{(0)}} \sum_{j=M^{(0)}+1}^{M} \lambda^{(J)}$$

where, $M^{(0)}$ is a burn-in period.

Step:Bayesian estimates of the parameter λ under LINEX loss function ,

$$\hat{\lambda}_{BL} = \frac{-1}{h} \ln\left[\frac{1}{M-M^{(0)}} \sum_{J=M^{(0)}+1}^{M} e^{-h\lambda^{(J)}}\right], h \neq 0,$$

Step 10: A $100(1-\gamma)\%$ Bayesian of λ can be obtained from the $\left(\frac{\gamma}{2}\right)$ and $1-\frac{\gamma}{2})$ sample quantiles of the empirical posterior pdf of MCMC draws

5 Monte Carlo Simulations

To evaluate the effectiveness of the GIED's parameters α and θ developed in the previous sections, in this section, we will perform different Monte Carlo simulations.

5.1 Simulation scenarios

To examine the behavior of the suggested point and interval estimators of α and θ obtained by maximum likelihood and Bayesian estimation methods, from GIED(2,1) population, we replicate the GHCS-I 2,000 times. All proposed comprehensive Monte Carlo simulations are conducted based on different combinations of *n*(total sample size) and *T*(threshold point) such as *n*(=30,50,80) and *T*(=1.5,2.5). Additionally, for each setting level of *n* and *T*, different choices of *k*,*m*(effective sample sizes) are also considered.

In frequentist examination, from the 1,000 GHCS-I samples, the MLEs as well as 95% ACIs (obtained from NA and NL methods) of α and θ are computed via 'maxLik' package proposed by Henningsen and Toomet(2011). In Bayesian analysis, to highlight the effects of the priors, two informative sets of hyper-parameters are used; namely:

- Prior-1: $(a_1,a_2) = (10,5)$ and $b_1 = b_2 = 5$;
- Prior-2: $(a_1,a_2) = (20,10)$ and $b_1 = b_2 = 10$.

It should be noted here that all specified hyper-parameter values of a_i and b_i for i = 1,2, associated with the unknown parameters α and θ are chosen in such a way that the prior average is equal to the expected value of the corresponding unknown parameter; see Kundu (2008). It is important to mention here, if there is no prior information about the parameters of interest, that the likelihood method may be better than the Bayes method because the latter is computationally more expensive. Via the proposed Metropolis-Hastings algorithm sampler, we generate 12,000 MCMC samples of α and θ from their conditional posterior distributions, and then the first 2,000 variates are ignored as burn-in. Then, using the remaining 10,000 MCMC samples, the desired computations of the proposed Bayes MCMC estimates (from SE and LINEX (for $h(= -2,+2)$) loss functions) and 95% BCI/HPD interval estimates of α and θ are obtained via the 'coda' package proposed by Plummer et al.(2006). All computations are performed using R 4.2.2 software by using two mainly statistical packages called 'maxLik' and 'coda'. Specifically, the average estimates (Av.Es) of α and θ are given by

$$\text{Av.E}(\alpha) = \frac{1}{2000}\sum_{i=1}^{2000} \breve{\alpha}^{(i)} \quad \text{and} \quad \text{Av.E}(\theta) = \frac{1}{2000}\sum_{i=1}^{2000} \breve{\theta}^{(i)}$$

respectively, where $\breve{\alpha}^{(i)}$ (as an example) is the estimate of α at ith sample.

The root mean squared errors (RMSEs) and mean relative absolute biases (MRABs) for all point estimates of α and θ are given, respectively, by

$$\mathrm{RMSE}(\breve{\alpha}) = \sqrt{\frac{1}{2000}\sum_{i=1}^{2000}\left(\breve{\alpha}^{(i)} - \alpha\right)^2} \quad \text{and} \quad \mathrm{RMSE}(\breve{\theta}) = \sqrt{\frac{1}{2000}\sum_{i=1}^{2000}\left(\breve{\theta}^{(i)} - \theta\right)^2},$$

and

$$\mathrm{MRAB}(\breve{\alpha}) = \frac{1}{2000}\sum_{i=1}^{2000}\frac{\left|\breve{\alpha}^{(i)} - \alpha\right|}{\alpha} \quad \text{and} \quad \mathrm{MRAB}(\breve{\theta}) = \frac{1}{2000}\sum_{i=1}^{2000}\frac{\left|\breve{\theta}^{(i)} - \theta\right|}{\theta}$$

Moreover, to compare the acquired interval estimates of α and θ, we consider two criteria, namely: average interval lengths (AILs) and coverage percentages (CPs) as

$$\mathrm{AIL}^{95\%}(\alpha) = \frac{1}{2000}\sum_{i=1}^{2000}\left(\mathcal{U}_{\breve{\alpha}^{(i)}} - \mathcal{L}_{\breve{\alpha}^{(i)}}\right) \quad \text{and} \quad \mathrm{AIL}^{95\%}(\theta) = \frac{1}{2000}\sum_{i=1}^{2000}\left(\mathcal{U}_{\breve{\theta}^{(i)}} - \mathcal{L}_{\breve{\theta}^{(i)}}\right)$$

and

$$\mathrm{CP}^{95\%}(\alpha) = \frac{1}{2000}\sum_{i=1}^{2000}\mathbf{1}^{\star}_{\left(\mathcal{L}_{\breve{\alpha}^{(i)}};\mathcal{U}_{\breve{\alpha}^{(i)}}\right)}(\alpha) \quad \text{and} \quad \mathrm{CP}^{95\%}(\theta) = \frac{1}{2000}\sum_{i=1}^{2000}\mathbf{1}^{\star}_{\left(\mathcal{L}_{\breve{\theta}^{(i)}};\mathcal{U}_{\breve{\theta}^{(i)}}\right)}(\theta)$$

respectively, where $\mathbf{1}^{\star}(\cdot)$ is an indicator, $(\mathcal{L}(\cdot),\mathcal{U}(\cdot))$ denotes the (lower,upper) interval limits of parameter.

5.2 Simulation discussions

In Tables 1-2, the Av.Es, RMSEs, and MRABs of α and θ are reported. On the other hand, in Tables 3-4, the AILs and CPs of α and θ are provided.

From Tables 1-4, in terms of lowest RMSE, MRAB, and AIL values and highest CP values, we provide the following observations:

- All offered estimates of α or θ have displayed satisfactory behavior.

- As n increases, all point (or interval) estimates of α or θ operate effectively, produce superior results, and hold the consistency feature. So, to get more accurate estimates, practitioners tend to increase the size of n appropriately.
- Comparing the proposed point estimation methodologies, due to the Bayes estimates being expressed using gamma density priors, the Bayes (point/interval) estimates using the Metropolis-Hastings procedure outperformed well the point and interval estimated developed from the maximum likelihood approach.
- Comparing the proposed loss functions, it is clear that the estimates produced by the LINEX loss of α or θ are overestimates when $h < 0$; they are also underestimates when $h > 0$, and both perform better compared to those developed from the SE loss. • Due to the fact that the variance of Prior-2 is less than the variance associated with Prior-1, it is noticeable that Bayesian point (or credible interval) estimates based on Prior-2 are superior to Prior-1 for all unknown parameters.
- Comparing the proposed interval estimation approaches, it is clear that:
  - The asymptotic interval estimates of α or θ constructed by ACI-NA showed better behavior than those constructed by its competitive ACI-NL method.
  - The credible interval estimates of α or θ constructed by HPD interval method showed better behavior than those constructed by its competitive BCI method.

- As T increases, it can be noted that:
  - The RMSEs and MRABs for all maximum likelihood (or Bayes' MCMC) estimates of α tend to increase while those of θ tend to decrease.
  - The AILs for ACI-NA (or ACI-NL) estimates of α increased while those obtained from BCI (or HPD interval) method decreased.
  - The AILs for all asymptotic (or credible) interval estimates of θ decreased.
  - The CPs for ACI-NA (or ACI-NL) estimates of α decreased while those obtained from BCI (or HPD interval) method increased.
  - The CPs for all asymptotic (or credible) interval estimates of θ increased.
  - The CPs of HPD intervals (in most situations) are almost closely (or greater than) to the specified nominal level 95% compared to others.
- Lastly, in the context of data acquired through the generalized type-I hybrid censored plan, we recommend using the Bayes frameworks with Metropolis-Hastings sampling to evaluate the GIED parameters.

Table 1: The Av.Es (1st column), RMSEs (2nd column) and MRABs (3rd column) of α and θ when $T = 1.5$.

n	(k,m)	Par.	MLE			SE			LINEX ($h = -2$)			LINEX ($h = +2$)		
									Prior-1 Prior-2					
30	(10,15)	α	1.701	1.611	0.804	1.978	0.373	0.162	2.002	0.319	0.156	1.952	0.364	0.180
						1.898	0.228	0.051	1.922	0.084	0.039	1.845	0.198	0.077
		θ	1.945	0.947	0.945	1.855	0.918	0.856	0.931	0.350	0.266	1.032	0.232	0.213
						1.463	0.745	0.740	1.094	0.279	0.242	1.006	0.133	0.150
	(15,20)	α	1.445	1.231	0.606	1.967	0.259	0.126	1.992	0.196	0.098	1.936	0.218	0.102
						1.944	0.219	0.043	1.964	0.064	0.028	1.900	0.169	0.057
		θ	1.708	0.711	0.708	1.603	0.684	0.605	0.944	0.283	0.222	1.020	0.241	0.179
						1.508	0.540	0.548	1.098	0.211	0.185	0.989	0.087	0.092
	(20,25)	α	1.925	0.773	0.356	1.731	0.227	0.092	1.930	0.145	0.074	1.774	0.119	0.063
						1.969	0.190	0.034	1.988	0.035	0.012	1.680	0.073	0.037
		θ	1.646	0.647	0.646	1.586	0.612	0.589	0.996	0.261	0.202	1.057	0.247	0.109
						1.511	0.512	0.484	1.140	0.214	0.165	1.071	0.079	0.071
50	(10,20)	α	1.802	1.562	0.778	2.261	0.358	0.152	2.313	0.294	0.146	2.204	0.327	0.160
						1.958	0.220	0.047	1.985	0.054	0.023	1.894	0.183	0.060
		θ	1.787	0.787	0.787	1.739	0.776	0.739	0.832	0.313	0.251	1.078	0.224	0.138
						1.740	0.679	0.680	1.141	0.184	0.141	1.051	0.063	0.060
	(20,30)	α	1.789	1.205	0.597	2.095	0.255	0.122	2.136	0.137	0.068	1.940	0.150	0.057
						2.006	0.213	0.037	2.025	0.040	0.018	2.031	0.143	0.044
		θ	1.518	0.529	0.518	1.488	0.520	0.488	0.875	0.255	0.203	1.096	0.233	0.156
						1.456	0.457	0.456	1.165	0.196	0.163	1.066	0.078	0.076
	(30,40)	α	1.807	0.751	0.322	1.789	0.189	0.067	1.893	0.084	0.050	1.772	0.095	0.037
						1.933	0.170	0.032	1.953	0.027	0.011	1.805	0.030	0.015
		θ	0.947	0.281	0.223	1.112	0.269	0.151	1.095	0.201	0.149	1.087	0.114	0.087
						1.148	0.216	0.132	1.149	0.193	0.120	1.024	0.050	0.043
80	(10,30)	α	1.532	1.365	0.664	2.036	0.350	0.136	1.968	0.219	0.101	2.087	0.229	0.113
						2.005	0.216	0.045	2.032	0.046	0.021	1.943	0.170	0.058
		θ	1.757	0.759	0.757	1.602	0.724	0.602	0.679	0.257	0.162	1.213	0.214	0.070
						1.160	0.467	0.463	0.799	0.165	0.110	1.132	0.031	0.025
	(30,50)	α	1.831	1.045	0.496	1.676	0.248	0.100	1.707	0.127	0.062	1.880	0.145	0.055
						1.944	0.208	0.036	1.971	0.039	0.014	1.641	0.095	0.037
		θ	1.382	0.418	0.412	1.412	0.413	0.382	1.107	0.236	0.126	0.858	0.209	0.121
						1.355	0.355	0.355	1.163	0.182	0.107	1.082	0.029	0.026
	(50,70)	α	1.526	0.564	0.252	2.081	0.176	0.062	2.008	0.052	0.024	1.905	0.064	0.032
						1.977	0.162	0.030	2.123	0.019	0.004	2.007	0.023	0.009
		θ	1.223	0.222	0.161	1.144	0.212	0.142	1.120	0.176	0.102	1.109	0.093	0.076
						1.148	0.189	0.071	0.959	0.149	0.097	1.035	0.035	0.024

Table 2: The Av.Es (1st column), RMSEs (2nd column) and MRABs (3rd column) of α and θ when $T = 2.5$.

n	(k,m)	Par.	MLE			SE			LINEX ($h = -2$)			LINEX ($h = +2$)		
									Prior-1					
									Prior-2					
30	(10,15)	α	1.501	1.707	0.853	1.728	0.866	0.385	1.756	0.639	0.316	1.697	0.817	0.455
						1.885	0.230	0.057	1.908	0.096	0.046	1.835	0.202	0.082
		θ	1.934	0.937	0.934	1.843	0.901	0.843	0.710	0.343	0.244	1.216	0.226	0.206
						1.734	0.744	0.734	1.175	0.252	0.176	1.150	0.125	0.132
	(15,20)	α	1.861	1.398	0.696	1.600	0.454	0.205	1.644	0.365	0.178	1.559	0.444	0.221
						1.922	0.222	0.044	1.941	0.072	0.033	1.882	0.172	0.059
		θ	1.674	0.675	0.674	1.599	0.665	0.600	0.762	0.281	0.212	1.165	0.203	0.165
						1.514	0.517	0.514	1.104	0.205	0.121	1.092	0.082	0.082
	(20,25)	α	1.935	0.936	0.457	1.230	0.255	0.105	1.368	0.160	0.076	1.090	0.182	0.075
						1.927	0.198	0.037	1.945	0.037	0.017	1.895	0.150	0.054
		θ	1.646	0.627	0.605	1.586	0.572	0.579	1.057	0.247	0.190	1.140	0.214	0.094
						1.511	0.512	0.511	0.917	0.183	0.132	1.068	0.074	0.068
50	(10,20)	α	1.780	1.668	0.833	2.084	0.590	0.280	2.111	0.527	0.262	2.053	0.592	0.295
						1.953	0.227	0.049	1.979	0.075	0.036	1.892	0.186	0.065
		θ	1.619	0.773	0.724	1.724	0.724	0.658	0.788	0.307	0.231	1.138	0.173	0.085
						1.453	0.492	0.488	1.084	0.180	0.103	1.060	0.055	0.051
	(20,30)	α	1.607	1.363	0.680	1.762	0.355	0.143	1.821	0.248	0.122	1.686	0.318	0.157
						1.979	0.218	0.043	2.004	0.049	0.025	1.924	0.154	0.058
		θ	1.456	0.494	0.456	1.489	0.489	0.481	0.826	0.280	0.187	1.164	0.189	0.107
						1.417	0.418	0.417	1.023	0.171	0.115	1.076	0.069	0.066
	(30,40)	α	1.640	0.818	0.379	1.439	0.244	0.085	1.475	0.095	0.057	1.873	0.143	0.053
						1.910	0.190	0.036	1.929	0.028	0.012	1.898	0.128	0.050
		θ	1.099	0.237	0.177	0.865	0.212	0.148	1.122	0.172	0.122	1.081	0.092	0.085
						1.150	0.176	0.130	0.902	0.157	0.100	1.043	0.044	0.035
80	(10,30)	α	1.270	1.511	0.750	2.036	0.486	0.222	2.087	0.405	0.202	1.968	0.483	0.241
						2.005	0.224	0.048	2.032	0.063	0.029	1.943	0.174	0.064
		θ	1.711	0.715	0.711	1.596	0.692	0.596	0.678	0.255	0.158	1.034	0.207	0.067
						1.488	0.464	0.453	1.091	0.156	0.090	1.009	0.030	0.023
	(30,50)	α	1.872	1.241	0.612	1.557	0.323	0.126	1.596	0.182	0.090	1.518	0.304	0.151
						1.931	0.212	0.039	1.957	0.046	0.021	1.869	0.145	0.055
		θ	1.279	0.296	0.289	1.289	0.289	0.279	0.759	0.217	0.122	1.098	0.178	0.087
						1.269	0.270	0.269	1.115	0.144	0.084	0.994	0.027	0.024
	(50,70)	α	1.344	0.688	0.329	1.915	0.194	0.075	1.955	0.055	0.029	1.409	0.116	0.040
						1.967	0.179	0.034	1.995	0.022	0.008	1.850	0.065	0.028
		θ	1.107	0.218	0.134	1.067	0.209	0.125	1.086	0.168	0.095	1.076	0.084	0.073
						1.060	0.168	0.068	1.107	0.117	0.085	1.024	0.031	0.022

Table 3: The AILs (1st column) and CPs (2nd column) of 95% asymptotic and credible intervals of α and θ when T = 1.5.

n	(*k*,*m*)	Par.	ACI-NA		BCI			
			ACI-NL		HPD			
					Prior-1		Prior-2	
30	(10,15)	α	2.241	0.892	0.866	0.922	0.540	0.931
			2.451	0.886	0.851	0.923	0.483	0.933
		θ	1.223	0.912	1.124	0.916	0.580	0.933
			1.244	0.911	1.117	0.917	0.532	0.936
	(15,20)	α	1.398	0.910	0.802	0.925	0.487	0.934
			1.561	0.904	0.749	0.928	0.436	0.936
		θ	1.113	0.915	1.109	0.918	0.575	0.934
			1.179	0.914	1.088	0.919	0.516	0.937
	(20,25)	α	0.822	0.918	0.636	0.932	0.442	0.936
			0.858	0.917	0.613	0.933	0.398	0.939
		θ	1.097	0.917	1.080	0.919	0.567	0.934
			1.134	0.915	1.034	0.920	0.508	0.938
50	(10,20)	α	1.682	0.903	0.834	0.924	0.522	0.932
			1.883	0.895	0.820	0.925	0.461	0.935
		θ	0.994	0.919	0.937	0.921	0.560	0.935
			1.036	0.917	0.920	0.922	0.501	0.938
	(20,30)	α	1.236	0.912	0.741	0.927	0.476	0.934
			1.409	0.907	0.731	0.928	0.421	0.936
		θ	0.879	0.922	0.798	0.925	0.554	0.935
			0.917	0.919	0.785	0.926	0.485	0.939
	(30,40)	α	0.713	0.921	0.565	0.934	0.431	0.937
			0.795	0.919	0.559	0.935	0.388	0.941
		θ	0.852	0.923	0.700	0.928	0.517	0.938
			0.893	0.921	0.675	0.930	0.466	0.941
80	(10,30)	α	1.526	0.906	0.819	0.925	0.503	0.933
			1.685	0.900	0.805	0.926	0.445	0.935
		θ	0.776	0.927	0.689	0.929	0.498	0.939
			0.798	0.925	0.657	0.932	0.444	0.943
	(30,50)	α	1.045	0.915	0.670	0.930	0.457	0.935
			1.125	0.913	0.655	0.931	0.406	0.938
		θ	0.670	0.930	0.657	0.932	0.486	0.940
			0.688	0.928	0.634	0.933	0.438	0.944
	(50,70)	α	0.652	0.924	0.528	0.936	0.403	0.939
			0.686	0.922	0.499	0.937	0.324	0.943
		θ	0.622	0.933	0.612	0.934	0.456	0.941
			0.633	0.930	0.585	0.936	0.429	0.945

Table 4: The AILs (1st column) and CPs (2nd column) of 95% asymptotic and credible intervals of α and θ when T = 2.5.

n	(*k*,*m*)	Par.	ACI-NA		BCI			
			ACI-NL		HPD			
					Prior-1		Prior-2	
30	(10,15)	α	1.178	0.908	1.025	0.916	0.545	0.930
			1.272	0.906	0.937	0.918	0.520	0.931
		θ	1.127	0.914	1.150	0.918	0.547	0.935
			1.228	0.913	1.115	0.919	0.516	0.937
	(15,20)	α	1.013	0.912	0.847	0.922	0.495	0.932
			1.037	0.911	0.822	0.923	0.443	0.935
		θ	0.974	0.918	1.105	0.920	0.565	0.936
			1.032	0.916	1.080	0.921	0.507	0.938
	(20,25)	α	0.659	0.922	0.646	0.931	0.451	0.935
			0.682	0.921	0.624	0.931	0.411	0.937
		θ	0.921	0.919	1.051	0.921	0.553	0.936
			0.988	0.918	1.001	0.922	0.502	0.938
50	(10,20)	α	1.107	0.910	0.918	0.918	0.529	0.931
			1.154	0.908	0.895	0.921	0.470	0.933
		θ	0.904	0.921	0.981	0.924	0.548	0.937
			0.941	0.919	0.917	0.924	0.486	0.940
	(20,30)	α	0.817	0.917	0.775	0.926	0.480	0.932
			0.876	0.915	0.761	0.927	0.428	0.936
		θ	0.761	0.925	0.937	0.926	0.522	0.938
			0.794	0.923	0.765	0.929	0.458	0.942
	(30,40)	α	0.644	0.923	0.576	0.934	0.442	0.936
			0.661	0.922	0.569	0.934	0.406	0.938
		θ	0.730	0.926	0.657	0.931	0.495	0.939
			0.766	0.925	0.644	0.933	0.449	0.943
80	(10,30)	α	1.066	0.911	0.869	0.921	0.509	0.931
			1.109	0.910	0.851	0.922	0.464	0.934
		θ	0.692	0.929	0.611	0.932	0.475	0.942
			0.710	0.927	0.573	0.935	0.429	0.945
	(30,50)	α	0.729	0.919	0.689	0.930	0.472	0.934
			0.749	0.916	0.674	0.930	0.416	0.937
		θ	0.584	0.933	0.521	0.935	0.452	0.942
			0.599	0.932	0.518	0.936	0.417	0.945
	(50,70)	α	0.586	0.928	0.555	0.935	0.409	0.939
			0.597	0.927	0.519	0.936	0.378	0.941
		θ	0.529	0.935	0.498	0.936	0.424	0.943
			0.536	0.933	0.479	0.938	0.385	0.946

6 Cancer Data Analysis

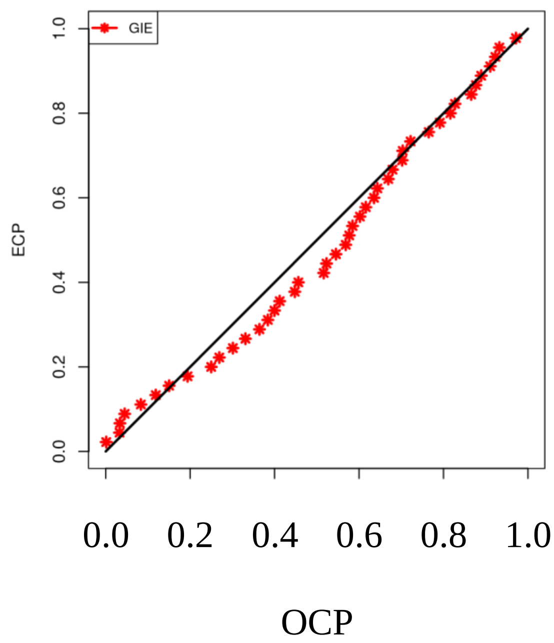
To show the adaptability of methodologies proposed to a real-life situation, in this section, a real-life data set gathered from the clinical sector is discussed. Now, we shall examine all proposed theoretical results of α and θ based on a cancer data set that represents the survival times for 44 patients suffering from head and neck cancer (HNC) disease and treated using radiotherapy and chemotherapy. In Table 5, the complete survival time points of the HNC data set are presented. This data set was first discussed by Efron (1988).

Table 5: Survival times (in days) of 44 HNC patients.

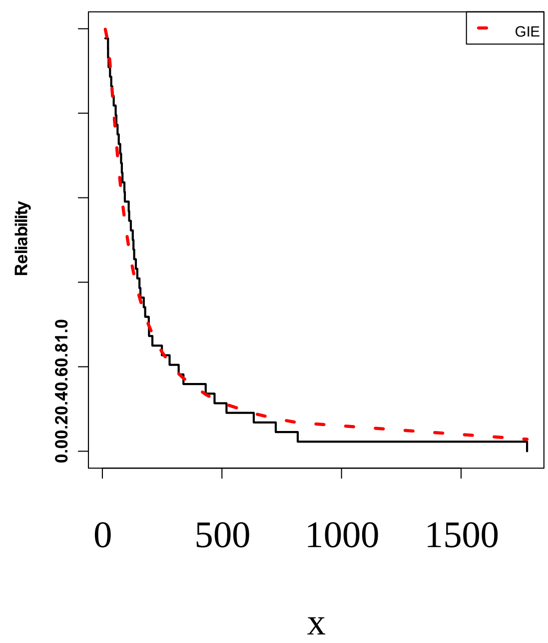
12.20, 23.56, 23.74, 25.87, 31.98, 37.00, 41.35, 47.38, 55.46,
58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84.00, 92.00, 94.00,
110.0, 112.0, 119.0, 127.0, 130.0, 133.0, 140.0, 146.0, 155.0,
159.0, 173.0, 179.0, 194.0, 195.0, 209.0,
249.0, 281.0, 319.0, 339.0, 432.0, 469.0, 519.0, 633.0, 725.0, 817.0, 1776

Before proceeding, we first need to see if the proposed GIED lifetime distribution is appropriate to fit the HNC data set or not. To achieve this, the MLEs $\hat{\alpha}$ and $\hat{\theta}$ are obtained and used to compute the Kolmogorov-Smirnov (K-S) distance and its associated *p*-value. As a result, the MLEs (along with their standard-errors (St.Ers)) of α and θ are 1.1677(0.2432) and 84.844(16.504), respectively, while the K-S(*p*-value) becomes

0.1072(0.6538). It is clear that the estimated p-value is far away from the pre-specified significance level; therefore, we have evidence to conclude that the GIED is a proper lifetime model to fit the HNC data.



(a) PP



(b) Reliability

Figure 3: The estimated/empirical PP and reliability plots of the GIE model from the HNC data set.

Additionally, from the entire HNC data, two goodness-of-fit diagrams are plotted and shown in Figure 3, namely: estimated/empirical probability-probability (PP) and estimated/empirical reliability function. All sup-plots displayed in Figure 3 support the same fitting result. Again, using the complete HNC data, the contour plot of the log-likelihood function is displayed in Figure 4 in turn to show the existence and uniqueness of the offered MLEs $\hat{\alpha}$ and $\hat{\theta}$. The maximum of the log-likelihood contour is denoted by point-x in the innermost. Figure 4 shows that the MLEs $\hat{\alpha}$ and $\hat{\theta}$ existed and are unique. For forthcoming statistical computations, we recommend considering the estimates $\hat{\alpha}$ ~= 1.1677 and $\hat{\theta}$ ~= 84.844 as suitable starting points.

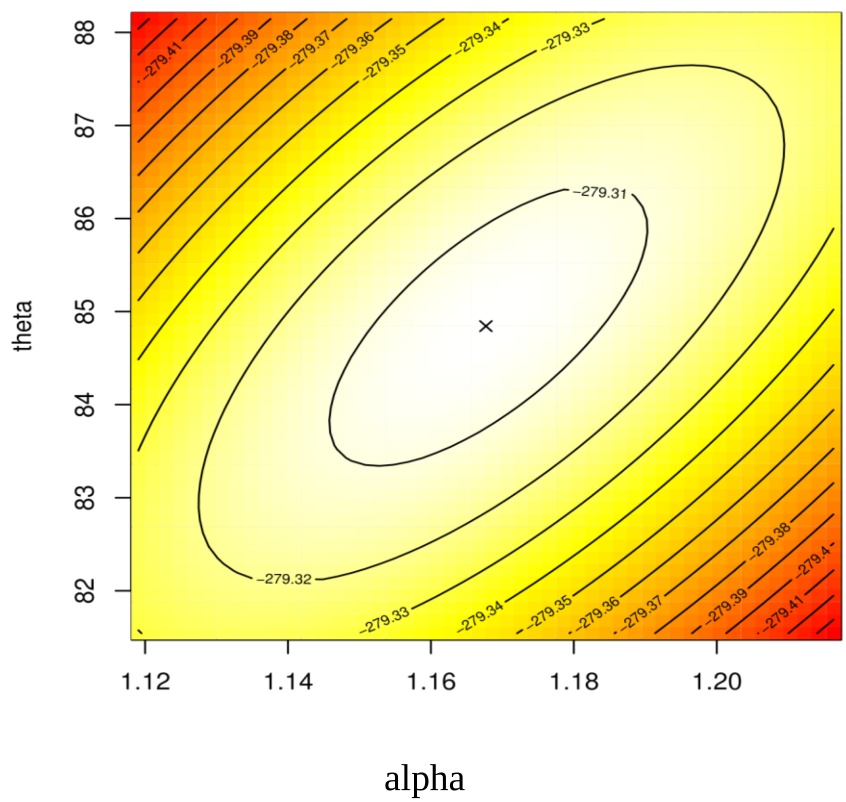


Figure 4: The contour plot of the GIE model from HNC data set.

Now, from the original HNC data set, three artificial GHCS-I samples (based on (k,m) =

(10,30) and different choices of T) are generated; see Table 6. For S_i for i = 1,2,3, the maximum likelihood and Bayes estimates (in addition to their St.Ers) as well as the asymptotic and credible interval estimates (in addition to their interval lengths (ILs)) of α and θ are computed and presented in Tables 7-8, respectively. Because we don't have any information about the GIED parameters α and θ from the HNC data set, we use the improper gamma priors by setting $a_i = 0$ and $b_i = 0$ for i = 1,2,3. Due to computation logic, the hyper-parameter values a_i and b_i for i = 1,2,3 are putted to be 0.001. Via the proposed Metropolis-Hastings algorithm, to carry out the Bayes' point (or credible) estimates, we generate 50,000 iterations and then discard the first 10,000 iterations as burn-in. Using the remaining 40,000 MCMC iterations, the Bayes' point estimates are evaluated using SE and LINEX (h(=-3,-0.03,+3)) loss functions.

Table 6: Three artificial GHCS-I samples from HNC data.

Sample	$T(d)$	Data
S_1	50(8)	12.2, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36
S_2	115(20)	12.2, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112
S_3	180(30)	12.2, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179

In terms of the lowest St.Er values, Table 7 indicates that the acquired Bayes MCMC estimates
(developed from the LINEX loss) of α and θ performed superiorly compared to those developed from the SE loss, while both outperformed those generated from the maximum likelihood method. On the other hand, in terms of the shortest IL values, Table 8 exhibits that the HPD interval estimates of α or θ perform well than those created by the BCI method, while both behave better than those obtained from the ACI-NA (or ACI-NL) approach. All facts provided in Tables 7-8 confirm the same simulation comments reported in Subsection 5.2.

Table 7: The point estimates of α and θ from HNC data.

Sample	Par.	MLE		SE		LINEX ($h=-3$) LINEX ($h=-0.03$) LINEX ($h=+3$)	
		Est.	St.Er	Est.	St.Er	Est.	St.Er
$\mathcal{S}_1$	α	0.5087	0.8736	0.4114	0.1307	0.4230	0.0857
						0.4115	0.0972
						0.4002	0.1085
	θ	54.037	42.273	53.888	0.1936	53.911	0.1262
						53.888	0.1487
						53.865	0.1721
$\mathcal{S}_2$	α	0.7074	0.8025	0.6132	0.1297	0.6253	0.0821
						0.6134	0.0941
						0.6014	0.1060
	θ	63.896	41.220	63.752	0.1862	63.773	0.1228
						63.752	0.1436
						63.731	0.1647
$\mathcal{S}_3$	α	0.9951	0.7962	0.8899	0.1429	0.9039	0.0912
						0.8900	0.1051
						0.8759	0.1192
	θ	76.927	25.462	76.783	0.1862	76.804	0.1230
						76.784	0.1438
						76.763	0.1648

Table 8: The 95% interval estimates of α and θ from HNC data.

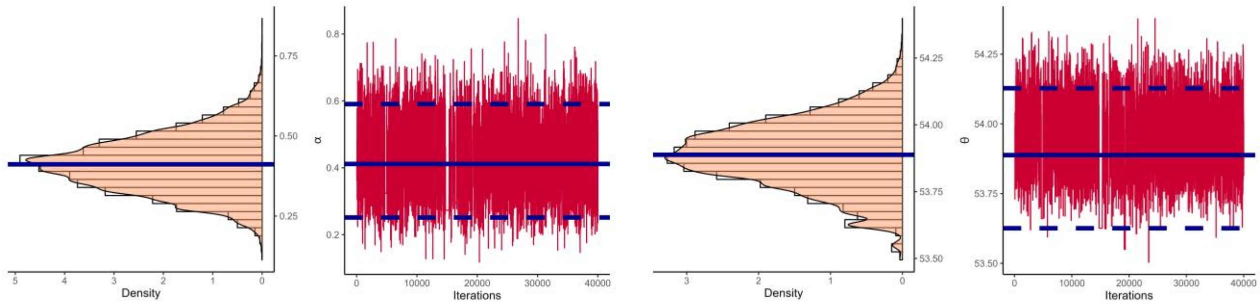
Sample	Par.	ACI-NA			BCI		
		ACI-NL			HPD		
		Low.	Upp.	IL	Low.	Upp.	IL
$\mathcal{S}_1$	α	0.2019	0.1130	0.9044	0.2513	0.5905	0.3392
		0.7913	0.2337	1.1073	0.2476	0.5853	0.3377
	θ	10.526	33.405	74.668	53.626	54.128	0.5020
		41.263	36.887	79.160	53.624	54.101	0.4773
$\mathcal{S}_2$	α	0.1951	0.3251	1.0898	0.4467	0.7940	0.3472
		0.7647	0.4120	1.2145	0.4464	0.7918	0.3454
	θ	10.341	43.627	84.164	63.525	63.985	0.4600
		40.537	46.527	87.747	63.534	63.993	0.4583
$\mathcal{S}_3$	α	0.1981	0.6070	1.3833	0.7008	1.0822	0.3814
		0.7763	0.6737	1.4699	0.6968	1.0705	0.3736
	θ	6.4663	64.254	89.601	76.560	77.016	0.4565
		25.348	65.243	90.705	76.566	77.019	0.4524

Moreover, some important properties for the MCMC iterations of α and θ, namely: mean, mode, 1st quartile (Q_1), 2nd quartile (Q_2), 3rd quartile (Q_3), standard deviation (St.D), and skewness (Skew.) are calculated and recorded in Table 9. As we anticipated, all results presented in Table 9 supported our findings shown in Table 7.

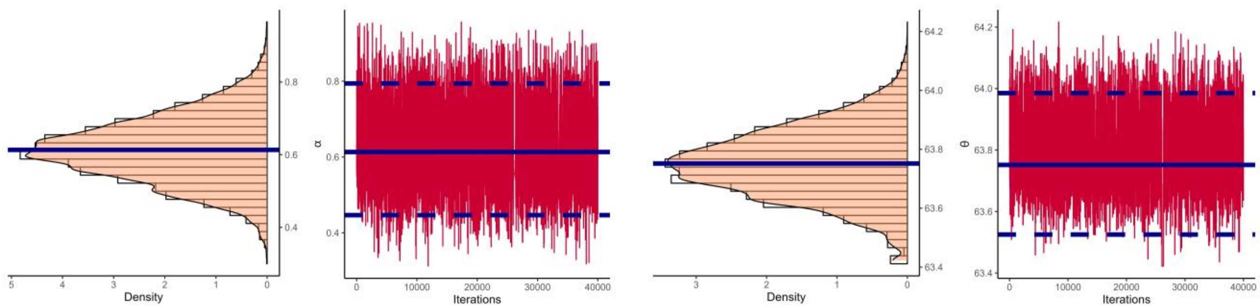
Table 9: Several statistics for 40,000 MCMC iterations of α and θ from HNC data.

Sample	Par.	Mean	Mode	Q_1	Q_2	Q_3	St.D	Skew.
$\mathcal{S}_1$	α	0.41136	0.27291	0.35024	0.40964	0.46920	0.08723	0.20229
	θ	53.8878	53.6256	53.8085	53.8893	53.9713	0.12372	-0.06508
$\mathcal{S}_2$	α	0.61324	0.55611	0.55454	0.61193	0.67123	0.08923	0.11404
	θ	63.7517	63.4225	63.6722	63.7504	63.8313	0.11825	0.03975
$\mathcal{S}_3$	α	0.88987	0.84384	0.82540	0.88932	0.95304	0.09660	0.04343
	θ	76.7835	76.4544	76.7041	76.7838	76.8630	0.11807	0.03598

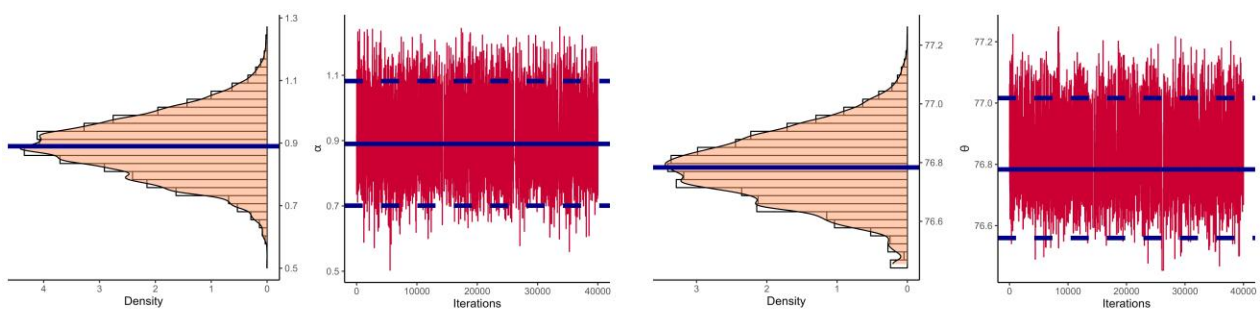
To demonstrate MCMC iteration convergence, S_i for i = 1,2,3, Figure 5 shows the density and trace plots of α and θ based on their staying 40,000 iterations. For specification, for each unknown parameter, the solid and dotted lines represent Bayes' estimate through the SE loss and BCI estimates, respectively. Figure 5 indicates that the MCMC technique works effectively, and the recommended burn-in sample size is effective. It indicates that the collected MCMC iterations of α or θ are relatively symmetrical. Moreover, all facts shown in Figure 5 support the same numerical findings reported in Table 9. Ultimately, the results of the proposed inference methodologies through the analysis of the HNC data furnish a good demonstration of the proposed generalized inverted exponential lifetime model.



(a) Sample S_1



(b) Sample S_2



(c) Sample S_3

Figure 5: The density (left) and trace (right) plots for 40,000 MCMC iterations of α and θ from HNC data.

7 Conclusion

In this paper, This study examined the estimation of the parameters of the generalized inverted exponential distribution under the GHCS-I. Maximum likelihood estimates and Bayesian estimates were derived using gamma priors, and the Markov Chain Monte Carlo method was applied for Bayesian inference.The results indicate that Bayesian estimates, particularly those obtained through the Metropolis-Hastings algorithm, outperformed the MLEs in terms of root mean square errors, mean relative absolute biases, and interval coverage probabilities. Bayesian estimates with higher prior precision demonstrated superior performance.Moreover, the LINEX loss function estimates provided more robust parameter estimates compared to the squared error loss function, especially under different censoring schemes. The analysis of a real cancer survival dataset confirmed the applicability and effectiveness of the proposed estimation methods. The generalized inverted exponential model provided a good fit to the data, as demonstrated by goodness-of-fit tests and parameter estimates. This validates the proposed methodology for lifetime data analysis under censoring schemes, making it a valuable approach for reliability and survival studies.

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