Statistical Inference on Simple Step-Stress Accelerated Life Testing for Gompertz Distribution Under Progressive Type-II Censoring

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Abstract:

We consider a simple step-stress model under the Gompertz distribution (GD) when the available data are type-II progressive censored. The cumulative exposure model is assumed when the lifetime of test units follows a Gompertz distribution. Maximum likelihood estimates and Bayes estimates are derived, utilizing the Markov Chain Monte Carlo (MCMC) method for computing Bayes estimates (BEs) and credible intervals. BEs of the parameters were obtained based on squared error (SE) and linear exponential (LINEX) loss functions under the assumptions of independent gamma priors. Finally, to illustrate these concepts, simulation studies and real-data examples are included.

Keywords: Step-stress accelerated life testing, Progressive type-II censoring, Maximum likelihood estimation, Bayes estimation, Gompertz distribution, Cumulative exposure model, Markov Chain Monte Carlo.

1 Introduction

Recently, accelerated life testing (ALT) has been extensively implemented in industries to obtain information about high-reliability, and adequate failure data in a compact time. The experiment or products are exposed to higher levels of stress to reduce testing time and cost. For some key references in the area of ALT with the cumulative exposure (CE) model, see, Miller and Nelson (1983) and Nelson (1990). The stress in ALT can be applied in three different ways, the most commonly used methods are step stress (SS), constant stress (CS), and progressive stress (PS).

To implement the step stress accelerated life testing (SSALT), a low stress to all units of the experiment is applied, and then a higher stress, hence, the stress applied on each unit is not constant but is gradually increased at a predetermined time, if only one change of the stress level is done, it is called a simple SS test. The simple SS models under the progressive type-II censoring scheme were discussed in the literature; by many authors, Wu and Lee (2005) obtained the maximum likelihood estimators (MLEs) of the unknown two parameters exponential distribution under simple SSALT with progressive type-II censoring by applying the CE model. They (2005) used the log-linear relationship of stress to estimate mean life from exponential lifetime distribution. Mohie El-Din et al. (2016) discussed the parametric inference of SSALT for the extension of exponential distribution under progressive type-II censoring with assumptions of the CE model. They obtained the MLEs of the unknown parameters and BEs under the squared error and linear exponential loss function. Mohie El-Din et al. (2021) obtained the MLEs and BEs of the unknown parameters for power generalized Weibull distribution under simple SSALT with progressive type-II censoring. They (2016) and (2021) introduced simulation studies to evaluate the performance of the estimators. Riad et al. (2021) studied SSALT for the Burr-XII distributions under the progressive type-II censoring scheme to calculate MLEs of the unknown parameters.

The paper is organized as follows: Section 2 a description of the lifetime model and the test assumptions. The MLEs of the parameters under simple SSALT with progressive type-II censoring from GD are obtained in Section 3. In Section 4, BEs of the unknown parameters SE and LINEX loss function are derived using MCMC. In Section 5, the simulation outcomes are represented. Finally, two real data sets are provided in Section 6.

2 Model Description

The Gompertz model was originally proposed by Benjamin Gompertz (1825) and it has been used as a growth model, especially in actuarial, human mortality, biomedical, and epidemiological studies. It is assumed that the lifetimes of the items being tested have GD with the probability density function (PDF), cumulative distribution function (CDF) and hazard rate function (HRF) as follows

$$f(x;\theta,\lambda)=\theta\lambda\ \exp(\lambda+\theta x-\lambda\ e^{\theta x}), \qquad x>0,\ \theta,\lambda>0, \qquad \textbf{(2.1)}$$

$$F(x;\theta,\lambda)=1-\exp(\lambda-\lambda e^{\theta x}), \qquad x>0,\ \theta,\lambda>0, \qquad \textbf{(2.2)}$$

$$h(x;\theta,\lambda)=\theta\lambda\ e^{\theta x}, \qquad x>0,\ \theta,\lambda>0, \qquad \textbf{(2.3)}$$

where θ and λ are the scale and shape parameters, respectively.

Suppose that the time data for failure comes from a CE model, we consider a simple SS model based on a progressive type-II censoring scheme with only two levels of stress, S_0 and S_1. Assume n identical units at an initial stress level S_0, r and $R_1, R_2, ..., R_r$ are fixed in advance. At a pre-fixed time τ, the stress level is changed to S_1. At the time of the first failure, R_1 of the $n-1$ surviving units are randomly removed from the experiment. At the time of the second failure, R_2 of the $n-2-R_1$ surviving units are randomly removed from the experiment, and so on. The life-testing experiment is terminated when the r^{th} time $x_{(r)}$ occurs at which time all the remaining $R_r = n-r-R_1-...-R_{r-1}$ surviving units are removed.

Progressively type-II censored data under simple SSALT are as follows

$$x_{(i)} = x_{(i:r:n)} = (x_{(1:r:n)} < ... < x_{(N_1:r:n)} < \tau < x_{(N_1+1:r:n)} < ... < x_{(r:r:n)}), \qquad \textbf{(2.4)}$$

where, N_1, number of units that fail before time τ at stress level S_0, and,

N_2, number of units that fail after time τ at stress level S_1.

The simple SSALT experiment is conducted under the following assumptions:

1- For any level of stress $S_i,\ i=0,1$, the lifetime distribution of a test unit is distributed as GD.

2- The relationship between the stress loading S and the life characteristic θ takes one of the following forms

- Exponential model: $\ln(\theta) = a + bS,$ where $a,b>0$ and S is a weathering variable.
- Inverse power model: $\ln(\theta) = a + b(\ln(S)),$ where $a,b>0$ and S is the voltage.

- Arrhenius model: $\ln(\theta) = a + b/-S,$ where $a, b > 0$ and S is the voltage.

Hence $\ln(\theta)$ is a linear function of the transformed stress $\mathrm{T}(S) = S, \ln(S)$ or $1/-S$ for the above three models. Furthermore, the relationship between the parameter θ_i and the stress level S_i is

$$\ln(\theta_i) = a + b\mathrm{T}_i, \qquad i = 1, 2, \tag{2.5}$$

where a and b are unknown parameters, and $\mathrm{T}_i = \mathrm{T}(S_i)$ is an increasing function of S, for more details on previous acceleration models, see Nelson (1990).

3- The shape parameter λ is constant for all stress levels.

4- The remaining life of a product depends only on its CE model, see Nelson (1990).

From the assumption of the CE model and the CDF given in (2.2), the CDF of a test unit under the simple SSALT is

$$G(x_{(i)}) = \begin{cases} F_1(x_{(i)}), & 0 \le x_{(i)} < \tau, \\ F_2(x_{(i)} + \tau - v), & \tau \le x_{(i)} < \infty, \end{cases} \tag{2.6}$$

where $F_j(x_{(i)}) = 1 - \exp(\lambda - \lambda e^{\theta_j x_{(i)}})$, for $j = 1, 2$, and v is the solution of $F_1(\tau) = F_2(v)$, therefore, the form of v is as follows

$$v = \frac{\theta_1}{\theta_2}\tau.$$

and the corresponding PDF is

$$g(x_{(i)}) = \begin{cases} f_1(x_{(i)}), & 0 \le x_{(i)} < \tau, \\ f_2(x_{(i)} + \tau - v), & \tau \le x_{(i)} < \infty, \end{cases} \tag{2.7}$$

where for $j = 1, 2$, $f_j(x_{(i)}) = \theta_j \lambda \exp(\lambda + \theta_j x_{(i)} - \lambda e^{\theta_j x_{(i)}})$.

3 Maximum Likelihood Estimation

In this section, the point and interval estimation of the unknown parameters for GD are obtained under simple SSALT with progressive type-II censoring. From the CDF in (2.6) and the corresponding PDF in (2.7), the likelihood function of the three-parameter λ, θ_1 and θ_2 based on observed progressively type-II censored data under simple SSALT given in (2.4) is obtained as

$$L(\lambda, \theta_1, \theta_2; x_{(i)}) = C\left(\prod_{i=1}^{N_1} f_1(x_{(i)})(1-F_1(x_{(i)}))^{R_i}\right)\left(\prod_{i=N_1+1}^{r} f_2(x_{(i)}+\tau-v)(1-F_2(x_{(i)}+\tau-v))^{R_i}\right),$$

(3.1) where $r = N_1 + N_2$ and $C = n(n-1-R_1)(n-2-R_1-R_2)\ldots(n-r+1-\sum_{i=1}^{r-1} R_i)$.

The MLEs of λ, θ_1 and θ_2 exist only in case at least one failure occurs before τ and at least one failure after τ, the log-likelihood function of (3.1) denoted by ℓ can be written as follows

$$\begin{aligned}\ell \propto r\ \log(\lambda) + N_1\ \log(\theta_1) + N_2\ \log(\theta_2) + \sum_{i=1}^{N_1} \theta_1 x_{(i)} + \sum_{i=N_1+1}^{r} \theta_2\ \omega(x_{(i)}) \\ + \sum_{i=1}^{N_1} (R_i+1)(\lambda - \lambda e^{\theta_1 x_{(i)}}) + \sum_{i=N_1+1}^{r} (R_i+1)\ (\lambda - \lambda e^{\theta_2\ \omega(x_{(i)})}),\end{aligned} \tag{3.2}$$

where $\omega(x_{(i)}) = x_{(i)} - \tau + \frac{\theta_1 \tau}{\theta_2}$.

The first partial derivatives of the log-likelihood function (3.2) with respect to λ, θ_1 and θ_2, are given respectively by

$$\frac{\partial \ell}{\partial \lambda} = \frac{r}{\lambda} + \sum_{i=1}^{N_1} (R_i+1)(1-e^{\theta_1 x_{(i)}}) + \sum_{i=N_1+1}^{r} (R_i+1)(1-e^{\theta_2\, \omega(x_{(i)})}), \tag{3.3}$$

$$\frac{\partial \ell}{\partial \theta_1} = \frac{N_1}{\theta_1} + \sum_{i=1}^{N_1} x_{(i)} + N_2\, \tau - \lambda \sum_{i=1}^{N_1} (R_i+1)\ (x_{(i)}\, e^{\theta_1 x_{(i)}}) - \lambda \tau \sum_{i=N_1+1}^{r} (R_i+1)\ e^{\theta_2\, \omega(x_{(i)})}, \tag{3.4}$$

and

$$\frac{\partial \ell}{\partial \theta_2}=\frac{N_2}{\theta_2}+\sum_{i=N_1+1}^{r}(x_{(i)}-\tau)-\lambda\sum_{i=N_1+1}^{r}(R_i+1)\ (x_{(i)}-\tau)\,e^{\theta_2\,\omega(x_{(i)})}, \qquad \textbf{(3.5)}$$

It looks impossible to obtain an exact solution of the above non-linear equations (3.3), (3.4), and (3.5) when equating to zero. So, we use an iterative technique to solve the previous nonlinear equations simultaneously to obtain $\hat{\lambda},\hat{\theta}_1$ and $\hat{\theta}_2$.

We derive the confidence intervals (CIs) of the unknown parameters based on the asymptotic variance-covariance matrix for MLEs of the elements of the vector of parameters $\varpi=(\lambda,\theta_1,\theta_2)$. The approximate asymptotic variance-covariance matrix is obtained by inverting Fisher information matrix $I_0(\varpi)$, practically, estimating $I_0^{-1}(\varpi)$ by $I_0^{-1}(\hat{\varpi})$

$$I_0^{-1}(\hat{\varpi})\cong\begin{bmatrix} -\dfrac{\partial^2\ell}{\partial\lambda^2} & -\dfrac{\partial^2\ell}{\partial\lambda\theta_1} & -\dfrac{\partial^2\ell}{\partial\lambda\theta_2} \\ -\dfrac{\partial^2\ell}{\partial\lambda\theta_1} & -\dfrac{\partial^2\ell}{\partial\theta_1^2} & -\dfrac{\partial^2\ell}{\partial\theta_1\theta_2} \\ -\dfrac{\partial^2\ell}{\partial\lambda\theta_2} & -\dfrac{\partial^2\ell}{\partial\theta_2\theta_1} & -\dfrac{\partial^2\ell}{\partial\theta_2^2} \end{bmatrix}^{-1}_{(\varpi=\hat{\varpi})}\cong\begin{bmatrix} \operatorname{var}(\hat{\lambda}) & \operatorname{cov}(\hat{\lambda},\hat{\theta}_1) & \operatorname{cov}(\hat{\lambda},\hat{\theta}_2) \\ \operatorname{cov}(\hat{\lambda},\hat{\theta}_1) & \operatorname{var}(\hat{\theta}_1) & \operatorname{cov}(\hat{\theta}_1,\hat{\theta}_2) \\ \operatorname{cov}(\hat{\lambda},\hat{\theta}_2) & \operatorname{cov}(\hat{\theta}_2,\hat{\theta}_1) & \operatorname{var}(\hat{\theta}_2) \end{bmatrix}_{(\varpi=\hat{\varpi})} \qquad (3.6)$$

Using (3.2), the second derivatives with respect to λ, θ_1 and θ_2 are as follows

$$\frac{\partial^2\ell}{\partial\lambda^2}=-\frac{r}{\lambda^2}$$

$$\frac{\partial^2\ell}{\partial\theta_1^2}=-\frac{N_1}{\theta_1^2}-\lambda\sum_{i=1}^{N_1}(R_i+1)\ (x_{(i)}^2 e^{\theta_1 x_{(i)}})-\lambda\tau^2\sum_{i=N_1+1}^{r}(R_i+1)\ e^{\theta_2\,\omega(x_{(i)})},$$

$$\frac{\partial^2\ell}{\partial\theta_2^2}=-\frac{N_2}{\theta_2^2}-\lambda\sum_{i=N_1+1}^{r}(R_i+1)(x_{(i)}-\tau)^2\ \ e^{\theta_2\,\omega(x_{(i)})},$$

$$\frac{\partial^2 \ell}{\partial \lambda \theta_1} = -\sum_{i=1}^{N_1} (R_i + 1)\ (x_{(i)}\, e^{\theta_1 x_{(i)}}) - \tau \sum_{i=N_1+1}^{r} (R_i + 1)\ \ e^{\theta_2\, \omega(x_{(i)})},$$

$$\frac{\partial^2 \ell}{\partial \lambda \theta_2} = -\sum_{i=N_1+1}^{r} (R_i + 1)\ e^{\theta_2\, \omega(x_{(i)})} (x_{(i)} - \tau)\,,$$

and

$$\frac{\partial^2 \ell}{\partial \theta_1 \theta_2} = -\lambda \tau \sum_{i=N_1+1}^{r} (R_i + 1)\ \ e^{\theta_2\, \omega(x_{(i)})}\ \ (x_{(i)} - \tau)$$

The CIs of unknown parameters $\varpi = (\lambda, \theta_1, \theta_2)$ are obtained based on the asymptotic normality of the MLEs. Thus, the 100(1- α)% two-sided CIs for the three-parameters λ, θ_1 and θ_2 are respectively, given by

$$\hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{Var(\hat{\lambda})} \qquad \text{and} \qquad \hat{\theta}_i \pm Z_{\frac{\alpha}{2}} \sqrt{Var(\hat{\theta}_i)}\ \ , \qquad i = 1,2,$$

where $Z_{\frac{\alpha}{2}}$ is the percentile of the standard normal distribution with right-tail probability $\frac{\alpha}{2}$.

4 Bayes Estimation

Bayesian inference for the unknown parameters of Gompertz model under simple SSALT with progressive type-II censoring using SE and LINEX loss functions are derived. Many researchers studying the GD, often assume the prior probability density function of the parameter follows a gamma distribution because the gamma prior is wealthy enough to cover the prior belief of the experimenter. Assume the model parameters λ, θ_1 and θ_2 are independent with priors as

$$\pi(\lambda) \propto \lambda^{\mu_1-1} e^{-\lambda a_1}, \qquad \lambda, \mu_1, a_1 > 0,$$
$$\pi(\theta_1) \propto \theta_1^{\mu_2-1} e^{-\theta_1 a_2}, \qquad \theta_1, \mu_2, a_2 > 0, \qquad \mathbf{(4.1)}$$
$$\pi(\theta_2) \propto \theta_2^{\mu_3-1} e^{-\theta_2 a_3}, \qquad \theta_2, \mu_3, a_3 > 0,$$

where all hyper-parameters μ_i and a_i, i =1,2,3, are assumed to be nonnegative and known. From (4.1), the joint prior density of parameters λ, θ_1 and θ_2 is given by

$$\pi(\lambda, \theta_1, \theta_2) \propto \lambda^{\mu_1-1} \theta_1^{\mu_2-1} \theta_2^{\mu_3-1} e^{-(\lambda a_1 + \theta_1 a_2 + \theta_2 a_3)}, \qquad \lambda, \theta_1, \theta_2 > 0 \qquad \mathbf{(4.2)}$$

The joint posterior density of λ, θ_1 and θ_2 is generated by using the likelihood function (3.1) and the joint prior (4.2) as follows

$$\begin{aligned}\pi(\lambda, \theta_1, \theta_2 \,|\, x) \propto\ & \lambda^{r+\mu_1-1} \theta_1^{N_1+\mu_2-1} \theta_2^{N_2+\mu_3-1} e^{-(\lambda a_1 + \theta_1 a_2 + \theta_2 a_3)} \\ & \times \prod_{i=1}^{N_1} \exp((R_i+1)(\lambda - \lambda e^{\theta_1 x_{(i)}}) + \theta_1 x_{(i)}) \\ & \times \prod_{i=N_1+1}^{r} \exp((R_i+1)\ (\lambda - \lambda e^{\theta_2 \omega(x_{(i)})}) + \theta_2\, \omega(x_{(i)}))\end{aligned} \qquad \mathbf{(4.3)}$$

BEs of any function of λ, θ_1 and θ_2 say $\zeta(\varpi) = \zeta(\lambda, \theta_1, \theta_2)$ under SE and LINEX loss functions can be obtained, respectively, as follows

$$\hat{\zeta}_{SE} = \int_{\varpi} \zeta(\lambda, \theta_1, \theta_2)\ L(\lambda, \theta_1, \theta_2; x_{(i)})\ \pi(\lambda, \theta_1, \theta_2) \quad d\varpi, \qquad (4.4)$$

and

$$\hat{\zeta}_{LINEX} = -\frac{1}{h} \ln\left(\int_{\varpi} \exp(-h(\zeta(\varpi)))\, L(\lambda, \theta_1, \theta_2; x_{(i)})\ \pi(\lambda, \theta_1, \theta_2)\ \ d\varpi\right), \qquad (4.5)$$

where $h \neq 0$ represents the shape parameter of the LINEX loss function. Obviously, the three integrals given by Equations (4.4) and (4.5) cannot be calculated analytically, so, one may utilize the MCMC. The MCMC method can be used to generate samples from the joint posterior density distribution (4.3) and in turn to compute the BEs of λ, θ_1 and θ_2, and the corresponding credible intervals under simple SSALT with progressive type-II censoring. Based upon the target posterior distribution (4.3). The conditional posterior distributions of λ, θ_1 and θ_2 have the following forms

$$\phi_1^*(\lambda|\theta_1,\theta_2) \propto \lambda^{r+\mu_1-1} \; e^{-\lambda[a_1+\sum_{i=1}^{N_1}(R_i+1)(1-e^{\theta_1 x_{(i)}})+\sum_{i=N_1+1}^{r}(R_i+1)(1-e^{\theta_2\,\omega(x_{(i)})})]},$$
$$\lambda \sim Gamma\left(r+\mu_1, a_1+\sum_{i=1}^{N_1}(R_i+1)(1-e^{\theta_1 x_{(i)}})+\sum_{i=N_1+1}^{r}(R_i+1)(1-e^{\theta_2\,\omega(x_{(i)})})\right) \quad \textbf{(4.6)}$$

$$\phi_2^*(\theta_1|\lambda,\theta_2) \propto \theta_1^{N_1+\mu_2-1}\, e^{-\theta_1 a_2}\prod_{i=1}^{N_1}\exp((R_i+1)\;(\lambda-\lambda e^{\theta_1 x_{(i)}})+(\theta_1\, x_{(i)}))$$
$$\times\prod_{i=N_1+1}^{r}\exp((R_i+1)\;(\lambda-\lambda e^{\theta_2\,\omega(x_{(i)})})+\theta_2\,\omega(x_{(i)})), \quad \textbf{(4.7)}$$

$$\phi_3^*(\theta_2|\lambda,\theta_1) \propto \theta_2^{N_2+\mu_3-1}\, e^{-\theta_2 a_3}\prod_{i=N_1+1}^{r}\exp((R_i+1)\;(\lambda-\lambda e^{\theta_2\,\omega(x_{(i)})})+\theta_2\,\omega(x_{(i)})) \quad \textbf{(4.8)}$$

From (4.6), the unknown parameter λ has gamma density, thus, samples of λ can be easily generated using a gamma-generating routine. But, the conditional posterior distributions (4.7) and (4.8) are very difficult to reduce analytically to known distributions, as a result, to derive from these distributions, one may employ the Metropolis-Hastings (M-H) algorithm with normal proposal distribution. For more details concerning the application of M-H, see Robert and Casella (2004). The detailed procedure of Gibbs within M-H algorithm to generate samples can be described as

Algorithm

Step 1: Start with an initial value $\lambda^{(0)}, \theta_1^{(0)}$ and $\theta_2^{(0)}$.

Step 2: Set t =1.

Step 3: Generate $\lambda^{(t)}$ from equation (4.6).

Step 4: Generate $\theta_1^{(t)}$ and $\theta_2^{(t)}$ using the M-H algorithm with proposal distribution

$q(\theta_1) = N(\theta_1^{(t-1)}, V(\theta_1))$ and $q(\theta_2) = N(\theta_2^{(t-1)}, V(\theta_2))$, respectively

- Generate θ_1^* from $N(\theta_1^{(t-1)}, V(\theta_1))$ and θ_2^* from $N(\theta_2^{(t-1)}, V(\theta_2))$.
- Evaluate the acceptance probabilities by

$$r_{\theta_1}(\theta_1^*, \theta_1^{(t-1)}) = \min\left[1, \frac{f(\theta_1^*)\ q(\theta_1^{(t-1)})}{f(\theta_1^{(t-1)})\ q(\theta_1^*)}\right],$$

and

$$r_{\theta_2}(\theta_2^*, \theta_2^{(t-1)}) = \min\left[1, \frac{f(\theta_2^*)\ q(\theta_2^{(t-1)})}{f(\theta_2^{(t-1)})\ q(\theta_2^*)}\right]$$

- Generate samples U_1 and U_2 from uniform (0,1) distribution.

Step 5: Set $t = t + 1$.

Step 6: Repeat steps 3-5, M times, and obtain $\zeta^{(t)} = (\lambda^{(t)}, \theta_1^{(t)}, \theta_2^{(t)})$, $t = 1,2, \ldots, M$.

Step 7: Under the SE loss function, obtain the Bayes estimates $\zeta = (\lambda, \theta_1, \theta_2)$

$$\tilde{\zeta}_{SE} = \frac{\sum_{t=M_0+1}^{M} \zeta^{(t)}}{M - M_0},$$

Step 8: Under the LINEX loss function, obtain the Bayes estimates of $\zeta = (\lambda, \theta_1, \theta_2)$

$$\tilde{\zeta}_{LINEX} = -\frac{1}{h}\ln\left(\sum_{t=M_0+1}^{M} \exp(-h\zeta^{(t)})/(M - M_0)\right),$$

where M_0 represents the number of burn-in samples.

Step 9: To obtain the credible intervals of $\zeta = (\lambda, \theta_1, \theta_2)$, order the MCMC sample of $\zeta^{(t)}$,

$t = 1,2,\ldots,M$ as $(\lambda^{(1)},\lambda^{(2)},...,\lambda^{(M)})$, $(\theta_1^{(1)},\theta_1^{(2)},...,\theta_1^{(M)})$, and $(\theta_2^{(1)},\theta_2^{(2)},...,\theta_2^{(M)})$. Then,

the 100(1-α)% symmetric credible interval of ζ can be obtained as

$$(\zeta^{((\alpha/2)(M-M_0))},\zeta^{((1-(\alpha/2))(M-M_0))}).$$

5 Simulation Study

To compare the behavior of the proposed point and interval estimators of the GD parameters λ,θ_1 and θ_2 under simple SSALT with progressive type-II censoring extensive Monte Carlo simulation studies are conducted based on several combinations of τ , n , r and $R_i, i=1,2,...,r$. The progressive type-II censoring with a simple SSALT mechanism is replicated 1000 times when the true value of $(\lambda,\theta_1,\theta_2)$ is taken as (1,0.5,1.5). Assuming τ $=(0.5,0.8)$ and $n=(40,80)$, the failure percentages (FPs) are taken as $(r/n\times 100\%)=((50,75)100\%)$ a specific amount r of each n.

Moreover, for each set of (n,r), three different progressive censoring (PC) designs are considered, namely

PC[1]: $(n-r,0^*(r-1))$,

PC[2]: $\left(0^*\left(\frac{r}{2}-1\right),n-r,0^*\left(\frac{r}{2}\right)\right)$,

and

PC[3]: $\left(0^*\left(r-1\right),n-r\right)$,

where $0^*\left(r-1\right)$ means 0 repeats $\left(r-1\right)$ times. To distinguish, the proposed PC[i] for $i=1,2,3$, represent the left, middle, and right progressive censoring plans, respectively.

To implement the experiment based on the philosophy sampling of progressive type-II censoring with a simple SSALT mechanism from the proposed GD, after assigning the values of τ , n , r and $R_i, i=1,2,...,r$, do the following steps proposed by Balakrishnan and Sandhu (1995)

Step 1: Set the true values of, λ, θ_1 and θ_2 .

Step 2: Put the predetermined values of τ , n , r and R_i, $i=1,2,...,r$.

Step 3: Generate a simple random sample $(U_1, U_2, ..., U_r)$ of size r from

Uniform (0,1)

distribution.

Step 4: Set $\rho_i = U_i^{(i+\sum_{j=r-i+1}^{r} R_j)^{-1}}$, for $i=1,2,...,r$.

Step 5: Obtain a progressive type-II censored sample $U_i^* = \prod_{j=r-i+1}^{r} \rho_j$, $i=1,2,...,r$.

Step 6: Find N_1 at time τ , such that $U_{N_1}^* < F_1(\tau) \le U_{N_1+1}^*$.

Step 7: Collect the order statistics $(x_{(1:r:n)}, ..., x_{(N_1:r:n)}, x_{(N_1+1:r:n)}, ..., x_{(r:r:n)})$ as follows

$$x_{(i:r:n)} = \begin{cases} \dfrac{1}{\theta_1} \log(1 - \dfrac{1}{\lambda} \log(1 - U_i^*)), & \text{for} \quad i = 1, 2, ..., N_1, \\ \dfrac{1}{\theta_2} \log(1 - \dfrac{1}{\lambda} \log(1 - U_i^*)) + \tau(1 - \dfrac{\theta_1}{\theta_2}), & \text{for} \quad i = N_1 + 1, ..., r. \end{cases}$$

Step 8: Use outputs in Step (7) to calculate the desired estimators.

Step 9: Redo Steps (3-8) 1000 times.

Once 1000 samples of progressive type-II censoring with simple SSALT are collected via R 4.2.2 programming software, as recommended by Nassar et al. (2024), we install two recommended packages

- A **'maxLik'** package (by Henningsen and Toomet (2011)) to evaluate the MLEs and 95% ACIs of λ, θ_1, and θ_2.
- A **'coda'** package (by Plummer et al. (2006)) to calculate the Bayes and BCIs of λ, θ_1 and θ_2.

In the Bayes model, the choice of the hyperparameters is the main issue. Thus, to see the effects of the prior distributions on the BEs and the associated BCIs estimators, two different sets of hyperparameters for each set of λ, θ_1 and θ_2 are utilized, namely

- Prior-1: $(\mu_1, \mu_2, \mu_3) = (5, 2.5, 7.5)$ and $a_i = 5$ for $i = 1, 2, 3$,
- Prior-2: $(\mu_1, \mu_2, \mu_3) = (10, 5, 15)$ and $a_i = 10$ for $i = 1, 2, 3$.

It is clear that the values of the hyperparameters of the unknown parameters λ, θ_1 or θ_2 are chosen in such a way that the prior mean becomes the expected value of the corresponding parameter, for more detail, see Kundu (2008).

To develop the Bayesian MCMC computations, following the sampling steps of M-H algorithm 12000 MCMC samples are generated and discard the first 2000 values as 'burn-in'. Hence, from the remaining 10000 MCMC samples, the average Bayes MCMC estimates are evaluated based on the SE and LINEX (with $h(-2, +2)$) loss functions. For each simulation setup, to run the MCMC sampler, the calculated MLE values of λ, θ_1 and θ_2 are used as initial values. Comparison between the proposed point estimates is made based on their estimated root mean squared errors (RMSEs) and mean absolute biases (MABs) values, whereas the comparison between the proposed interval estimates their average interval lengths (AILs) and coverage percentages (CPs).

Now, the average estimates (AEs), RMSEs, MABs, AILs, and CPs of λ (as an example) are calculated using the following formulas, respectively, as

$$\text{AE}(\tilde{\lambda})=\frac{1}{1000}\sum_{i=1}^{1000}\tilde{\lambda}^{(i)}, \qquad \text{RMSE}(\tilde{\lambda})=\sqrt{\frac{1}{1000}\sum_{i=1}^{1000}(\tilde{\lambda}^{(i)}-\lambda)^2},$$

$$\text{MAB}(\tilde{\lambda})=\frac{1}{1000}\sum_{i=1}^{1000}\left|\tilde{\lambda}^{(i)}-\lambda\right|, \qquad \text{AIL}_{(1\text{-}\alpha)\%}(\tilde{\lambda})=\frac{1}{1000}\sum_{i=1}^{1000}(U(\tilde{\lambda}^{(i)})-L(\tilde{\lambda}^{(i)})),$$

and

$$\text{CP}_{(1\text{-}\alpha)\%}(\tilde{\lambda})=\frac{1}{1000}\sum_{i=1}^{1000}\Phi(L(\tilde{\lambda}^{(i)}),U(\tilde{\lambda}^{(i)}))(\lambda),$$

where $\tilde{\lambda}^{(i)}$ denotes the classical or Bayes estimate obtained at the i^{th} sample of λ, $\Phi(.)$ represents the indicator function, and $(L(\tilde{\lambda}^{(i)}),U(\tilde{\lambda}^{(i)}))$ denotes (lower, upper) interval limits respectively of $(1-\alpha)\%$ ACI or BCI of λ. Similarly, the AE, RMSE, MAB, AIL, and CP values for the other unknown parameters θ_1 and θ_2 can be easily calculated.

In Tables (1-4), the AEs, RMSEs, and MABs of λ,θ_1 and θ_2 are reported. On the other hand, the AILs and CPs of λ,θ_1 and θ_2 are listed in Tables (5-8). In regard to the lowest RMSE, MAB, and AIL values, in addition to the highest CP values, we report the following observations:

- Generally, the acquired point and interval estimates of the unknown parameters of λ,θ_1 and θ_2 behave satisfactorily.
- As n (or r) increases, all estimates of λ,θ_1 and θ_2 perform better. A similar result is found when the total number of removal patterns $R_i, i=1,2,...,r$ decreases.

- Comparing the PC[i] for $i = 1,2,3$, it is noted that the unknown parameters θ_1 and θ_2 behave well based on PC[3] (when the remaining items $n-r$ removed at the last stage) as well of λ behave well based on PC[1] (when the remaining items $n-r$ removed at the first stage) than others.
- As τ increases, the RMSEs and MABs of all estimates of λ and θ_2 increase while those of θ_1 decrease.
- Comparing the gamma priors 1 and 2, since the variance of Prior-2 is less than the variance of Prior-1, it can be seen that the Bayes point estimations and BCI estimations of all unknown parameters outperformed based on Prior-2 compared to those developed from Prior-1, and both are better than those obtained from the MLE (or ACI) estimates.
- It is known that more accurate estimates will be obtained when the priors are used more accurately. Thus, for all settings, the MCMC estimates of λ, θ_1 or θ_2 provide more accurate results compared to those obtained from the likelihood method.
- The estimates developed by the LINEX function of λ, θ_1 or θ_2 are overestimates (when $h = -2$) and also underestimates (when $h = +2$). This result is due to the fact that Bayes findings based on the asymmetric LINEX loss function have greater flexibility due to form parameter loss than those developed under the symmetric SE loss function.
- The Bayes MCMC paradigm using the M-H algorithm to estimate the unknown parameters of GD under simple SSALT with progressive type-II censoring is recommended.

Table 1. The AEs (1st Col.), RMSEs (2nd Col.), and MABs (3rd Col.) when (n,τ) = (40,0.5).

PC	Par.	MLE			SE			LINEX ($h=-2$)			LINEX ($h=+2$)		
					Prior-1 Prior-2			Prior-1 Prior-2			Prior-1 Prior-2		
					(n,r) = (40,20)								
PC[1]	λ	1.8627	1.2648	0.9380	1.2357	0.6995	0.5606	1.2405	0.6464	0.5214	1.0521	0.3342	0.3293
					0.9373	0.4452	0.3567	1.1984	0.4216	0.3393	0.8706	0.2515	0.2455
	θ_1	0.9037	1.4563	0.8830	0.4359	0.5917	0.5910	0.5894	0.4975	0.4942	0.4258	0.4674	0.3860
					0.5335	0.5886	0.5457	0.5653	0.4527	0.4363	0.5446	0.3184	0.3134
	θ_2	1.5749	1.7181	1.2754	1.5940	0.8743	0.8703	1.6343	0.5020	0.4945	1.4174	0.4783	0.4630
					1.5321	0.7877	0.6213	1.8619	0.4948	0.4745	1.4593	0.3984	0.3794
PC[2]	λ	1.6899	1.4806	1.0977	1.2433	0.7778	0.6232	1.2554	0.7520	0.5344	1.1783	0.4828	0.4753
					1.0520	0.5028	0.3902	1.3293	0.4940	0.3700	0.9703	0.3031	0.2998
	θ_1	0.8254	1.4489	0.7750	0.4887	0.5378	0.5371	0.5595	0.4643	0.4406	0.3828	0.4418	0.3742
					0.4214	0.5166	0.4885	0.4451	0.4456	0.4290	0.4286	0.3101	0.3043
	θ_2	1.5128	1.3255	1.1159	1.7101	0.7001	0.6980	1.7536	0.4902	0.4684	1.5386	0.3827	0.3556
					1.6271	0.6673	0.5412	2.0035	0.4697	0.4599	1.6187	0.3770	0.3462
PC[3]	λ	2.2145	1.7471	1.1790	1.2956	0.9886	0.8437	1.3272	0.8125	0.5476	1.1183	0.6271	0.6190
					0.9440	0.6149	0.4342	1.1589	0.6072	0.4173	0.8540	0.3538	0.3515
	θ_1	0.7939	1.3647	0.7545	0.4367	0.4441	0.4146	0.5488	0.4207	0.3803	0.4290	0.3460	0.3441
					0.4810	0.4240	0.3960	0.5579	0.4050	0.3571	0.5382	0.2835	0.2775
	θ_2	1.6869	1.2038	1.0565	1.7459	0.6205	0.6152	1.8029	0.3895	0.3756	1.6592	0.3556	0.3352
					1.6688	0.6034	0.5375	2.0832	0.3765	0.3549	1.6572	0.3283	0.3230
					(n,r) = (40,30)								
PC[1]	λ	1.7113	0.9912	0.6164	1.4509	0.5029	0.4442	1.6054	0.4050	0.3585	1.4190	0.2402	0.2139
					1.2490	0.3792	0.3248	1.6364	0.3681	0.3033	1.1346	0.2114	0.2066
	θ_1	0.7174	1.3139	0.7095	0.5391	0.4153	0.3177	0.5963	0.3391	0.2646	0.4902	0.2293	0.2286
					0.4991	0.3603	0.2860	0.5655	0.3141	0.2355	0.5189	0.1742	0.1714
	θ_2	1.5431	1.1584	0.9485	1.4611	0.5791	0.5551	1.5274	0.3755	0.3297	1.4465	0.3147	0.3106
					1.4770	0.5559	0.5120	1.5452	0.3617	0.3211	1.1662	0.3087	0.3029
PC[2]	λ	2.1260	1.1635	0.6730	1.3618	0.5521	0.4557	1.5439	0.4418	0.3625	1.2753	0.2643	0.2191
					1.3515	0.3864	0.3414	1.4480	0.3802	0.3258	1.2382	0.2185	0.2133
	θ_1	0.7797	1.1466	0.6662	0.4930	0.3461	0.2642	0.5720	0.3046	0.2080	0.4327	0.1660	0.1651
					0.4492	0.3296	0.2122	0.6051	0.2884	0.1997	0.6050	0.1494	0.1467

	θ_2	1.5920	1.0389	0.8291	1.5799	0.5353	0.5248	1.5961	0.3667	0.3107	1.2551	0.2734	0.2654
					1.5894	0.5269	0.4149	1.6145	0.3108	0.3001	1.1748	0.2558	0.2470
PC[3]	λ	1.9307	1.2223	0.7444	1.4841	0.6441	0.4915	1.4874	0.5545	0.4193	1.3293	0.3005	0.2836
					1.4762	0.3923	0.3451	1.4934	0.3853	0.3360	1.3174	0.2344	0.2282
	θ_1	0.8222	1.0358	0.6028	0.5216	0.3023	0.2253	0.6863	0.3124	0.1992	0.5033	0.1564	0.1540
					0.4781	0.2955	0.2032	0.6084	0.2751	0.1939	0.6714	0.1398	0.1385
	θ_2	1.5311	0.9535	0.7490	1.4914	0.4293	0.4860	1.6186	0.3624	0.2979	1.3703	0.2547	0.2475
					1.5927	0.3867	0.3714	1.6249	0.2829	0.2584	1.2311	0.2478	0.2261

Table 2. The AEs (1st Col.), RMSEs (2nd Col.), and MABs (3rd Col.) when (n,τ) = (80,0.5).

PC	Par.	MLE			SE			LINEX ($h = -2$)			LINEX ($h = +2$)		
					Prior-1 Prior-2			Prior-1 Prior-2			Prior-1 Prior-2		
						(n,r) = (80,40)							
PC[1]	λ	1.2744	0.5567	0.5277	1.3175	0.3971	0.3404	1.3194	0.3200	0.2218	1.2914	0.2135	0.1783
					1.1669	0.3283	0.2364	1.2707	0.3123	0.2015	0.9856	0.1581	0.1534
	θ_1	0.7840	1.0352	0.5496	0.4878	0.2705	0.2085	0.5371	0.2122	0.1913	0.4341	0.1525	0.1452
					0.5406	0.2514	0.1994	0.6872	0.2093	0.1841	0.5612	0.1115	0.1050
	θ_2	1.5949	0.9179	0.7382	1.6933	0.3986	0.3455	1.7343	0.3462	0.2868	1.4218	0.2458	0.2126
					1.6512	0.3609	0.3367	1.6925	0.2745	0.2453	1.4712	0.2128	0.1984
PC[2]	λ	1.2722	0.7801	0.5360	1.2568	0.4449	0.3549	1.4196	0.3893	0.2489	1.2428	0.2236	0.1870
					1.1392	0.3536	0.2381	1.2383	0.3296	0.2147	0.9590	0.1702	0.1638
	θ_1	0.6704	1.0380	0.5463	0.5144	0.2612	0.1975	0.5640	0.1926	0.1840	0.4788	0.1167	0.1166
					0.4942	0.2227	0.1885	0.6207	0.1854	0.1668	0.6043	0.0719	0.0714
	θ_2	1.5076	0.8844	0.7177	1.7149	0.3430	0.2842	1.7475	0.2877	0.2408	1.5249	0.1928	0.1834
					1.6822	0.3214	0.2683	1.7271	0.2348	0.2343	1.5571	0.1851	0.1783
PC[3]	λ	1.2733	1.0401	0.6832	1.2461	0.4687	0.4328	1.3008	0.3931	0.2819	1.2372	0.2362	0.2032
					1.0734	0.3581	0.2404	1.1974	0.3405	0.2255	0.9797	0.2040	0.2013
	θ_1	0.7853	0.9006	0.5253	0.4879	0.2583	0.1803	0.5739	0.1701	0.1323	0.4760	0.0925	0.0924
					0.4884	0.1914	0.1631	0.6029	0.1012	0.1009	0.5205	0.0562	0.0548
	θ_2	1.6076	0.8521	0.6779	1.9489	0.3193	0.2519	1.9945	0.2486	0.2137	1.5481	0.1863	0.1726
					1.8936	0.2926	0.2455	2.0369	0.1757	0.1822	1.8265	0.1567	0.1504
						(n,r) = (80,60)							
PC[1]	λ	1.4639	0.4699	0.4894	1.3670	0.2859	0.2452	1.4519	0.1836	0.1775	1.3091	0.1140	0.1186

					0.9290	0.2657	0.2013	1.0049	0.1638	0.1589	0.8571	0.0740	0.0759
	θ_1	0.6727	0.8783	0.5176	0.5301	0.2314	0.1702	0.5789	0.1153	0.1218	0.4622	0.0588	0.0530
					0.4845	0.1607	0.1275	0.5766	0.0877	0.0875	0.5455	0.0450	0.0446
	θ_2	1.5715	0.7743	0.6195	1.7107	0.2815	0.2401	1.7453	0.1835	0.1764	1.5919	0.1411	0.1232
					1.6276	0.2512	0.1834	1.7915	0.1521	0.1439	1.5575	0.1236	0.1157
PC[2]	λ	1.6276	0.5278	0.4386	1.1702	0.3676	0.2993	1.2587	0.2593	0.1830	1.1292	0.1569	0.1423
					0.9316	0.3074	0.2273	1.0975	0.1964	0.1696	0.9267	0.0955	0.0852
	θ_1	0.6639	0.7055	0.3907	0.5216	0.2177	0.1516	0.5712	0.1132	0.1106	0.4433	0.0416	0.0382
					0.6166	0.1549	0.1232	0.6614	0.0794	0.0793	0.6385	0.0364	0.0336
	θ_2	1.5917	0.6464	0.5209	1.8224	0.2810	0.2340	1.8549	0.1527	0.1512	1.6320	0.1361	0.1185
					1.7481	0.2453	0.1794	1.8311	0.1425	0.1343	1.5955	0.1120	0.0984
PC[3]	λ	1.5811	0.8798	0.5700	1.1284	0.3859	0.3311	1.3511	0.3002	0.2384	1.0789	0.2063	0.1585
					1.1188	0.3266	0.2337	1.2183	0.2199	0.1984	0.8671	0.1170	0.1064
	θ_1	0.7384	0.6161	0.3829	0.5164	0.1949	0.1497	0.5635	0.0902	0.0888	0.4536	0.0385	0.0330
					0.5924	0.1425	0.1191	0.6378	0.0405	0.0382	0.6467	0.0207	0.0187
	θ_2	1.5511	0.5972	0.4753	1.9288	0.2432	0.1953	1.9684	0.1168	0.1439	1.7439	0.1074	0.1066
					1.8459	0.2337	0.1722	1.9751	0.1098	0.1287	1.6918	0.0947	0.0861

Table 3. The AEs (1st Col.), RMSEs (2nd Col.), and MABs (3rd Col.) when (n,τ) = (40,0.8).

PC	Par.	MLE			SE			LINEX ($h=-2$)			LINEX ($h=+2$)		
					Prior-1			Prior-1			Prior-1		
					Prior-2			Prior-2			Prior-2		
						(n,r) = (40,20)							
PC[1]	λ	1.3512	1.6576	1.0897	1.1635	0.7237	0.5964	1.3518	0.7092	0.5381	1.1323	0.4147	0.4099
					1.1396	0.4570	0.3804	1.2990	0.4362	0.3772	0.9365	0.3435	0.3392
	θ_1	0.7176	0.9266	0.6662	0.6311	0.3917	0.3829	0.6834	0.3203	0.3196	0.5481	0.2883	0.1845
					0.4358	0.3831	0.3494	0.4644	0.3162	0.2416	0.4249	0.1406	0.1375
	θ_2	1.5177	1.7768	1.2892	1.6949	1.0325	1.0282	1.7398	0.5240	0.5101	2.1627	0.4940	0.4751
					1.9647	0.9037	0.7380	1.6166	0.5104	0.4820	1.7869	0.4039	0.3898
PC[2]	λ	1.8535	1.8558	1.1358	1.1917	0.8727	0.6391	1.2512	0.7854	0.5744	1.1843	0.5130	0.4760
					1.0474	0.5149	0.3931	1.2832	0.5165	0.3795	0.9672	0.3520	0.3458
	θ_1	0.6836	0.8968	0.6587	0.6253	0.3724	0.3615	0.6762	0.3041	0.2976	0.5014	0.2759	0.1827
					0.4796	0.3617	0.3278	0.4848	0.2982	0.2145	0.4695	0.1179	0.1151

	θ_2	1.2988	1.4343	1.1744	1.8081	0.7779	0.7754	1.8645	0.5196	0.4966	2.2754	0.4822	0.4379
					2.0437	0.7228	0.5831	1.7187	0.5018	0.4685	1.8400	0.3842	0.3540
PC[3]	λ	2.1563	2.0592	1.3516	0.9811	1.0734	0.8791	1.1599	0.8300	0.8106	0.9171	0.6395	0.6550
					1.0132	0.6288	0.4900	1.2343	0.6284	0.4448	1.0015	0.3904	0.3872
	θ_1	0.7171	0.8743	0.5986	0.5245	0.3317	0.2600	0.6243	0.2629	0.2106	0.4432	0.2114	0.1632
					0.4194	0.2840	0.2295	0.4264	0.2599	0.2015	0.4115	0.0940	0.0885
	θ_2	1.3644	1.2138	1.0814	1.9279	0.6648	0.6627	1.9966	0.5001	0.4795	2.5282	0.3769	0.3437
					2.2039	0.6388	0.5737	1.8118	0.4808	0.4635	1.9340	0.3630	0.3310
							(n,r) = (40,30)						
PC[1]	λ	1.9960	1.4672	0.6509	1.3865	0.5247	0.4558	1.4773	0.4992	0.4046	1.1976	0.2875	0.2154
					1.1079	0.3988	0.3424	1.1920	0.3799	0.3332	1.0282	0.2812	0.2103
	θ_1	0.6831	0.8444	0.5912	0.5122	0.3144	0.2153	0.5669	0.2251	0.1906	0.5042	0.1891	0.1622
					0.4709	0.2791	0.2060	0.6286	0.2068	0.1847	0.4682	0.0895	0.0868
	θ_2	1.5012	1.1858	1.0129	1.2758	0.6316	0.6273	1.3710	0.4379	0.4599	1.4716	0.3735	0.3362
					1.4293	0.6037	0.5279	1.1799	0.4018	0.4014	1.3598	0.3234	0.3201
PC[2]	λ	2.3925	1.3941	0.8901	1.2637	0.6542	0.4735	1.4778	0.5181	0.4324	1.1495	0.3246	0.2244
					1.1647	0.4182	0.3530	1.2025	0.4093	0.3421	1.1455	0.3159	0.2194
	θ_1	0.6602	0.7993	0.5313	0.4928	0.2896	0.2121	0.5438	0.2100	0.1834	0.4745	0.1624	0.1575
					0.4537	0.2732	0.2012	0.5961	0.2014	0.1676	0.4376	0.0828	0.0770
	θ_2	1.4829	1.0703	0.8688	1.3074	0.5822	0.5416	1.4039	0.4098	0.3645	1.5142	0.3619	0.3238
					1.4710	0.5499	0.4916	1.2084	0.3866	0.3511	1.3965	0.3185	0.3127
PC[3]	λ	1.9095	1.4758	1.0469	0.9775	0.7904	0.5775	1.1782	0.5622	0.4554	0.9689	0.3497	0.3292
					1.0465	0.4427	0.3989	1.1511	0.4242	0.3698	1.0133	0.3351	0.3130
	θ_1	0.7021	0.7953	0.5196	0.4908	0.2764	0.1813	0.6029	0.1933	0.1720	0.4643	0.1306	0.1299
					0.4534	0.2601	0.1789	0.6116	0.1729	0.1521	0.4188	0.0797	0.0762
	θ_2	1.4969	0.9950	0.8209	1.2879	0.5317	0.5034	1.3911	0.3843	0.3732	1.5571	0.3259	0.3130
					1.5135	0.5142	0.4758	1.1873	0.3321	0.3366	1.4374	0.3060	0.2933

Table 4. The AEs (1st Col.), RMSEs (2nd Col.), and MABs (3rd Col.) when (n,τ) = (80,0.8).

PC	Par.	MLE	SE	LINEX ($h=-2$)	LINEX ($h=+2$)
			Prior-1	Prior-1	Prior-1
			Prior-2	Prior-2	Prior-2
			(n,r) = (80,40)		

PC[1]	λ	1.3872	0.9289	0.5863	1.2486	0.4551	0.4079	1.3683	0.4147	0.3165	1.1835	0.2257	0.2130
					0.9755	0.3587	0.2668	1.1031	0.3299	0.2542	0.9736	0.1784	0.1741
	θ_1	0.6357	0.6739	0.5057	0.5269	0.2584	0.1757	0.6600	0.1647	0.1311	0.5071	0.1140	0.1071
					0.5118	0.1907	0.1734	0.5873	0.1315	0.1230	0.5070	0.0730	0.0696
	θ_2	1.5164	0.9695	0.7436	1.6972	0.4244	0.3823	1.7406	0.3827	0.3276	1.7313	0.2859	0.2864
					1.6751	0.4083	0.3723	1.6449	0.2958	0.2916	1.6231	0.2429	0.2608
PC[2]	λ	1.5310	1.0802	0.6062	0.9718	0.4857	0.4174	1.1309	0.4320	0.3354	0.9036	0.2749	0.2253
					0.9841	0.3743	0.2949	1.1554	0.3437	0.2919	0.8934	0.2261	0.2241
	θ_1	0.6574	0.6668	0.4816	0.5849	0.2438	0.1699	0.6292	0.1469	0.1225	0.5268	0.0621	0.0560
					0.5244	0.1802	0.1498	0.6349	0.0659	0.0580	0.5151	0.0551	0.0544
	θ_2	1.4564	0.9280	0.7327	1.6832	0.3742	0.3226	1.7252	0.2976	0.2756	1.7486	0.2414	0.2436
					1.6946	0.3432	0.2861	1.6373	0.2533	0.2541	1.6446	0.2158	0.2245
PC[3]	λ	1.8535	1.2454	0.7185	1.2852	0.5091	0.4431	1.2973	0.4376	0.3825	1.1877	0.2672	0.2384
					1.0881	0.3689	0.3100	1.1527	0.3591	0.3035	0.9700	0.2440	0.2098
	θ_1	0.6442	0.6290	0.4224	0.6812	0.2343	0.1646	0.7862	0.1426	0.1115	0.6685	0.0547	0.0511
					0.4976	0.1704	0.1425	0.6608	0.0580	0.0569	0.4754	0.0519	0.0492
	θ_2	1.5644	0.8661	0.7125	1.9543	0.3676	0.3079	2.0101	0.2585	0.2431	2.1273	0.2261	0.2231
					2.0582	0.2943	0.2725	1.8810	0.2359	0.2352	1.9820	0.1913	0.2040
(n,r) = (80,60)													
PC[1]	λ	1.7317	0.7216	0.5184	1.1881	0.3298	0.2819	1.2189	0.1945	0.1992	1.0904	0.1294	0.1287
					0.9792	0.2756	0.2046	0.9909	0.1861	0.1867	0.9783	0.0942	0.0980
	θ_1	0.6199	0.5727	0.4209	0.5525	0.2145	0.1540	0.5932	0.1240	0.1104	0.5357	0.0319	0.0430
					0.5195	0.1431	0.1384	0.5366	0.0502	0.0485	0.4994	0.0247	0.0232
	θ_2	1.5311	0.7952	0.6389	1.7992	0.3134	0.2605	1.8316	0.1892	0.1762	1.8823	0.1431	0.1312
					1.8342	0.2884	0.1963	1.7141	0.1533	0.1590	1.7296	0.1258	0.1213
PC[2]	λ	1.4529	0.8165	0.5069	1.2335	0.3823	0.3251	1.2567	0.2680	0.2400	1.1794	0.1744	0.1960
					1.1704	0.3129	0.2490	1.2156	0.2511	0.2133	0.9848	0.1327	0.1294
	θ_1	0.6146	0.5132	0.3743	0.6582	0.1998	0.1387	0.6906	0.0944	0.0932	0.6299	0.0317	0.0304
					0.5311	0.1323	0.1153	0.6024	0.0415	0.0377	0.5177	0.0175	0.0132
	θ_2	1.5401	0.6811	0.5536	1.9378	0.2938	0.2418	1.9795	0.1542	0.1545	2.0034	0.1375	0.1258
					1.9399	0.2654	0.1865	1.8538	0.1438	0.1389	1.8134	0.1153	0.1171
PC[3]	λ	1.5825	0.8821	0.5755	1.1878	0.4253	0.3638	1.2435	0.3214	0.2887	1.0708	0.2192	0.2149
					1.2137	0.3430	0.2524	1.2840	0.2837	0.2407	0.9985	0.1512	0.1460

θ_1	0.6141	0.5087	0.3565	0.4751	0.1645	0.1370	0.5038	0.0235	0.0199	0.4680	0.0215	0.0142
				0.4515	0.1278	0.1038	0.4876	0.0222	0.0195	0.4370	0.0166	0.0121
θ_2	1.4661	0.6271	0.5084	1.8629	0.2786	0.2273	1.9012	0.1423	0.1473	2.0216	0.1130	0.1131
				1.9535	0.2431	0.1756	1.7614	0.1249	0.1320	1.8133	0.1022	0.1051

Table 5. The AILs (1st Col.) and CPs (2nd Col.) when τ = 0.5.

(n,r)	PC	Par.	95% ACI		95% BCI			
					Prior-1		Prior-2	
(40,20)	PC[1]	λ	2.873	0.885	1.358	0.916	0.994	0.921
		θ_1	4.311	0.821	1.188	0.907	0.995	0.910
		θ_2	5.468	0.806	1.880	0.892	1.141	0.905
	PC[2]	λ	2.976	0.881	1.404	0.912	1.168	0.918
		θ_1	3.587	0.833	0.978	0.910	0.966	0.912
		θ_2	4.858	0.814	1.660	0.897	1.067	0.907
	PC[3]	λ	3.225	0.876	1.432	0.911	1.252	0.915
		θ_1	3.362	0.837	0.962	0.912	0.945	0.913
		θ_2	4.271	0.819	1.521	0.901	0.998	0.908
(40,30)	PC[1]	λ	1.973	0.896	1.113	0.921	0.977	0.924
		θ_1	3.231	0.839	0.943	0.915	0.877	0.919
		θ_2	4.127	0.821	1.149	0.906	0.980	0.909
	PC[2]	λ	2.185	0.892	1.164	0.918	1.106	0.920
		θ_1	3.013	0.841	0.930	0.916	0.867	0.920
		θ_2	3.889	0.824	0.994	0.908	0.937	0.912
	PC[3]	λ	2.437	0.888	1.245	0.914	1.150	0.917
		θ_1	2.743	0.845	0.860	0.919	0.825	0.922
		θ_2	3.537	0.828	0.990	0.908	0.901	0.913

Table 6. The AILs (1st Col.) and CPs (2nd Col.) when τ = 0.5.

(n,r)	PC	Par.	95% ACI		95% BCI			
					Prior-1		Prior-2	
(80,40)	PC[1]	λ	1.574	0.911	0.863	0.924	0.770	0.933
		θ_1	2.683	0.847	0.818	0.924	0.650	0.928
		θ_2	3.420	0.831	0.969	0.910	0.884	0.914

	PC[2]	λ	1.751	0.908	0.943	0.921	0.806	0.930
		θ_1	2.619	0.849	0.799	0.925	0.599	0.931
		θ_2	3.341	0.833	0.924	0.912	0.835	0.916
	PC[3]	λ	1.891	0.905	1.022	0.918	0.828	0.929
		θ_1	2.450	0.852	0.743	0.927	0.578	0.933
		θ_2	3.255	0.836	0.912	0.913	0.822	0.918
(80,60)	PC[1]	λ	1.115	0.919	0.799	0.927	0.562	0.939
		θ_1	2.309	0.854	0.636	0.932	0.541	0.935
		θ_2	2.930	0.838	0.879	0.915	0.810	0.918
	PC[2]	λ	1.238	0.917	0.815	0.925	0.597	0.936
		θ_1	1.984	0.858	0.548	0.936	0.515	0.938
		θ_2	2.498	0.842	0.835	0.916	0.788	0.919
	PC[3]	λ	1.326	0.915	0.935	0.921	0.629	0.933
		θ_1	1.901	0.860	0.538	0.937	0.468	0.941
		θ_2	2.403	0.843	0.805	0.917	0.774	0.920

Table 7. The AILs (1st Col.) and CPs (2nd Col.) when τ = 0.8.

(n,r)	PC	Par.	95% ACI		95% BCI			
					Prior-1		Prior-2	
(40,20)	PC[1]	λ	2.967	0.882	1.395	0.911	1.129	0.918
		θ_1	2.963	0.875	0.931	0.913	0.889	0.916
		θ_2	5.711	0.801	2.018	0.887	1.182	0.903
	PC[2]	λ	3.233	0.878	1.488	0.908	1.176	0.916
		θ_1	2.839	0.878	0.898	0.916	0.886	0.919
		θ_2	5.161	0.809	1.783	0.894	1.158	0.904
	PC[3]	λ	3.486	0.873	1.617	0.905	1.270	0.913
		θ_1	2.619	0.883	0.881	0.917	0.829	0.920
		θ_2	4.451	0.815	1.669	0.898	1.153	0.904
(40,30)	PC[1]	λ	2.130	0.892	1.135	0.917	1.072	0.921
		θ_1	2.536	0.884	0.773	0.921	0.599	0.926
		θ_2	4.199	0.819	1.187	0.904	1.129	0.906
	PC[2]	λ	2.360	0.889	1.201	0.914	1.151	0.917
		θ_1	2.406	0.887	0.756	0.922	0.582	0.927
		θ_2	4.051	0.822	1.061	0.906	1.049	0.907

	PC[3]	λ	2.843	0.883	1.306	0.910	1.187	0.916
		θ_1	2.396	0.888	0.725	0.924	0.566	0.929
		θ_2	3.714	0.826	1.031	0.906	0.995	0.909

Table 8. The AILs (1st Col.) and CPs (2nd Col.) when τ = 0.8.

(n,r)	PC	Par.	95% ACI		95% BCI			
					Prior-1		Prior-2	
(80,40)	PC[1]	λ	1.899	0.904	0.885	0.921	0.835	0.930
		θ_1	2.317	0.891	0.703	0.926	0.547	0.931
		θ_2	3.632	0.828	0.984	0.908	0.937	0.910
	PC[2]	λ	1.972	0.901	0.984	0.918	0.877	0.927
		θ_1	2.197	0.894	0.649	0.929	0.537	0.934
		θ_2	3.534	0.830	0.934	0.910	0.898	0.912
	PC[3]	λ	2.191	0.896	1.005	0.916	0.888	0.926
		θ_1	2.071	0.897	0.599	0.933	0.518	0.938
		θ_2	3.309	0.833	0.913	0.912	0.887	0.913
(80,60)	PC[1]	λ	1.235	0.917	0.801	0.926	0.595	0.937
		θ_1	2.023	0.902	0.537	0.937	0.490	0.939
		θ_2	3.115	0.836	0.903	0.912	0.873	0.914
	PC[2]	λ	1.366	0.914	0.898	0.922	0.626	0.934
		θ_1	1.778	0.910	0.502	0.938	0.464	0.941
		θ_2	2.664	0.839	0.899	0.913	0.846	0.916
	PC[3]	λ	1.483	0.912	0.975	0.918	0.638	0.930
		θ_1	1.762	0.911	0.488	0.941	0.457	0.942
		θ_2	2.524	0.841	0.860	0.914	0.782	0.919

6 Real-Life Applications

In this section, two real data sets are analyzed to illustrate the proposed simple SSALT model.

6.1 Set 1: Solar Lighting Device

In this application, we look at a data set representing two dominant failure levels (controller failure and capacitor failure) to evaluate the reliability features of a solar lighting device. The device's major failure mode is controller failure, and the stress factor is temperature, which is increased during the test from 293 to 353 K, with the common operating temperature of 293 K. The stress change time point of this data set is assigned to be 5 (in a hundred hours). This data set was originally proposed by Han and Kundu (2014) and reanalyzed later by Kotb and Mohie (2020). In Table (9), the number of failure time points under designated stress levels 1 and 2 is 16 and 15, respectively.

Table 9. Failure times of solar lighting device.

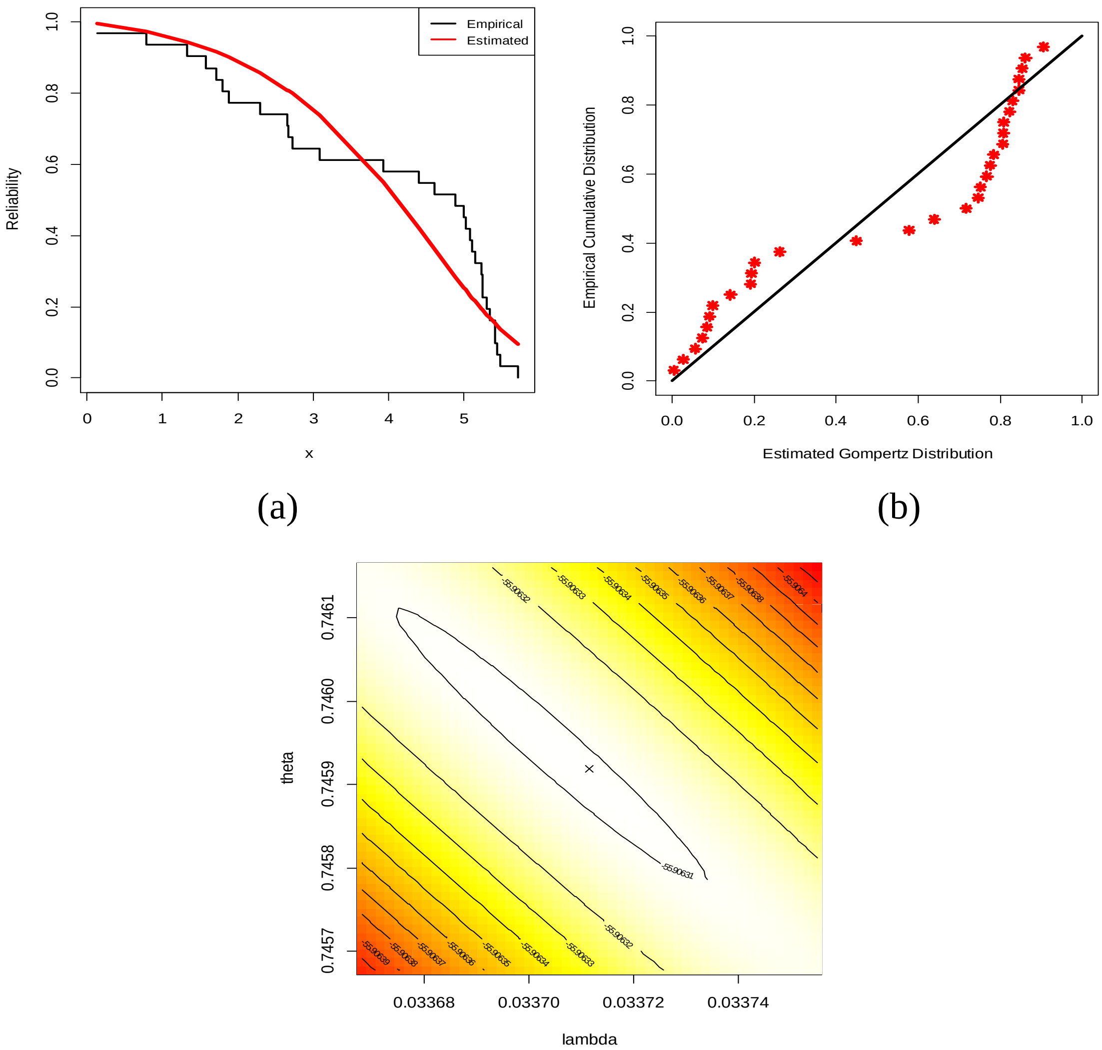
Level 1 ($\tau < 5$)
0.140, 0.783, 1.324, 1.582, 1.716, 1.794, 1.883, 2.293, 2.660, 2.674, 2.725, 3.085, 3.924, 4.396, 4.612, 4.892
Level 2 ($\tau > 5$)
5.002, 5.022, 5.082, 5.112, 5.147, 5.238, 5.244, 5.247, 5.305, 5.337, 5.407, 5.408, 5.445, 5.483, 5.717

To check whether or not the solar lighting device data set fits the GD, the Kolmogorov-Smirnov (KS) goodness of fit statistic (along with its *P*-value) is calculated. In Table (10), the MLEs (with their standard errors (Std.Ers)), 95% ACI estimates of GD parameters, and KS(*P*-value) are presented. Since the fitted *P*-value is greater than the significance level of 0.05, hence we cannot reject the null hypothesis, Table (10) shows that the GD fits the solar lighting device data satisfactorily.

Table 10. Fit results of the Gompertz distribution from solar lighting device data.

Par.	MLE		95% ACI			KS(*P*-value)
	Est.	Std.Er	Lower	Upper	Length	
λ	0.0337	0.0257	0.0000	0.0843	0.0843	0.0233 (0.058)
θ	0.7459	0.1422	0.4663	1.0238	0.5575	

Employing the complete solar lighting device data, Figure (1) depicts several plots, namely: estimated and empirical reliability lines, probability-probability (PP), and contour of the log-likelihood function. As a result, it supports the same KS outputs reported in Table (10) and shows that the calculated MLEs $\hat{\lambda} = 0.0337$ and $\hat{\theta} = 0.7459$ might exist and are unique.



(a) (b)

(c)

Figure 1. The fitted reliability line (a), PP (b), and contour (c) diagrams of the Gompertz lifetime model from solar lighting device data.

Now, to see the usefulness of the acquired point and interval estimators, three progressively type-II censored samples with a simple

SSALT are created from the entire solar lighting device data based on various choices of $R_i, i = 1, 2, ..., r$; see Table (11). So, for each generated sample, the point estimates (including the maximum likelihood and Bayes estimates) and the interval estimates (including the asymptotic and credible interval estimates) of λ, θ_1 and θ_2 are calculated; see Tables (12) and (13). Obviously, we do not have any prior information about λ, θ_1 and θ_2, thus, we set $\mu_i = \alpha_i = 0.001$ for $i = 1, 2, 3$, which means that the posterior density becomes quite close to the likelihood function. We also run the proposed MCMC procedure with a burn-in of 10000 followed by 40000 iterations. Thus, the Bayes point estimates through the SE and LINEX (for $h(=-3, -0.03, +3)$) loss functions are evaluated. For beginning our iterations, the initial values of λ, θ_1 and θ_2 are taken as, $\hat{\lambda}, \hat{\theta}_1$, and $\hat{\theta}_2$, respectively. It is clear, from Table (12), that both the classical and Bayesian estimates are very close to each other, while the latter performed better than the former with respect to the minimum standard errors. A similar behavior, as shown in Table (13), is also noted in the case of the interval estimates.

Table 11. Three simple step-stress samples from solar lighting device data when $(n, r) = (31, 20)$.

Sample	Scheme	Data
A	$(11, 0^{19})$	0.140, 1.324, 1.582, 1.716, 1.794, 2.293, 2.660, 2.674, 2.725, 3.924, 4.396, 4.892, 5.022, 5.082, 5.112, 5.238, 5.244, 5.305, 5.337, 5.407
B	$(0^{10}, 11, 0^{9})$	0.140, 0.783, 1.324, 1.582, 1.716, 1.794, 1.883, 2.293, 2.660, 2.674, 2.725, 3.085, 3.924, 4.612, 4.892, 5.082, 5.112, 5.238, 5.247, 5.305
C	$(0^{19}, 11)$	0.140, 0.783, 1.324, 1.582, 1.716, 1.794, 1.883, 2.293, 2.660, 2.674, 2.725, 3.085, 3.924, 4.396, 4.612, 4.892, 5.002, 5.022, 5.082, 5.112

Table 12. The point estimates (1st Col.) with their Std.Ers (2nd Col.) from solar lighting device data.

Sample	Par.	MLE		SEL		LINEX					
$h \rightarrow$						-3		-0.03		+3	
A	λ	0.3489	0.4048	0.3312	0.0490	0.3344	0.0145	0.3312	0.0176	0.3281	0.0208
	θ_1	0.2524	0.1712	0.2464	0.0339	0.2481	0.0043	0.2464	0.0059	0.2448	0.0076
	θ_2	2.6382	0.9456	2.6134	0.0558	2.6171	0.0211	2.6134	0.0248	2.6096	0.0286
B	λ	0.1881	0.1754	0.1805	0.0325	0.1820	0.0062	0.1805	0.0076	0.1791	0.0090
	θ_1	0.3813	0.1778	0.3741	0.0340	0.3758	0.0055	0.3741	0.0072	0.3725	0.0087
	θ_2	3.0712	1.0829	3.0550	0.0429	3.0574	0.0139	3.0550	0.0162	3.0527	0.0185
C	λ	0.5751	0.8684	0.5605	0.0418	0.5628	0.0123	0.5606	0.0145	0.5582	0.0169
	θ_1	0.1638	0.1728	0.1586	0.0241	0.1594	0.0044	0.1586	0.0053	0.1578	0.0061
	θ_2	1.8973	1.4141	1.8810	0.0431	1.8834	0.0139	1.8810	0.0163	1.8786	0.0187

Table 13. The interval estimates from solar lighting device data.

Sample	Par.	95% ACI			95% BCI		
		Lower	Upper	Length	Lower	Upper	Length
A	λ	0.0000	1.1423	1.1423	0.2435	0.4234	0.1799
	θ_1	0.0000	0.5879	0.5879	0.1823	0.3133	0.1310
	θ_2	0.7849	4.4915	3.7066	2.5192	2.7124	0.1932
B	λ	0.0000	0.5319	0.5319	0.1205	0.2441	0.1236
	θ_1	0.0329	0.7296	0.6968	0.3100	0.4399	0.1298
	θ_2	0.9488	5.1936	4.2448	2.9784	3.1332	0.1547
C	λ	0.0000	2.2771	2.2771	0.4839	0.6379	0.1540
	θ_1	0.0000	0.5026	0.5026	0.1143	0.2057	0.0914
	θ_2	0.0000	4.6690	4.6690	1.8045	1.9594	0.1550

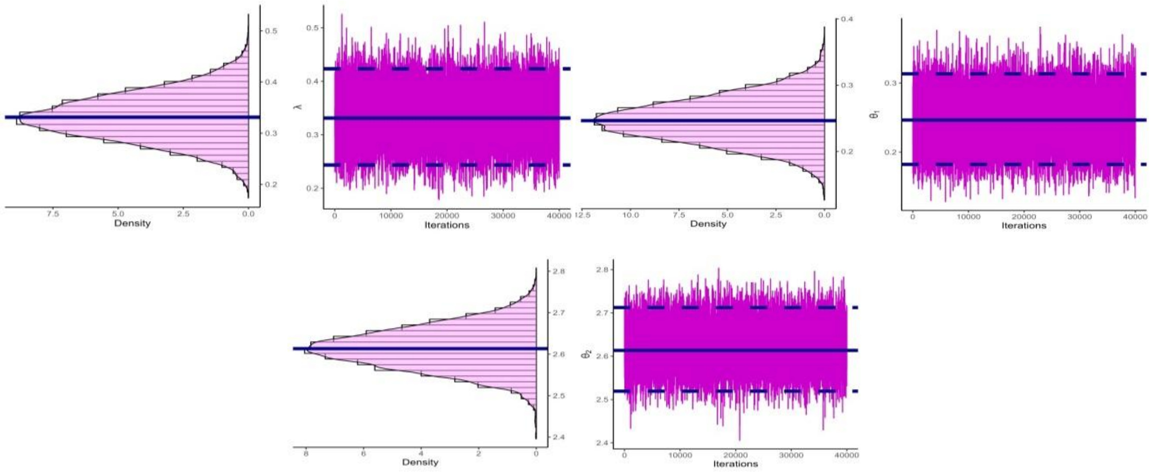
In Table (14), some useful properties of λ, θ_1 and θ_2 based on their staying 40000 MCMC draws, namely: mean, mode, three quartiles (say, ($Q_i,\ i=1,2,3$)), standard deviation (Std.D), and skewness (Sk.) are listed. It shows that the acquired MCMC estimates of λ, θ_1 or θ_2 are fairly symmetrical.

Table 14. Summary of MCMC outputs of λ, θ_1 and θ_2 from solar lighting device data.

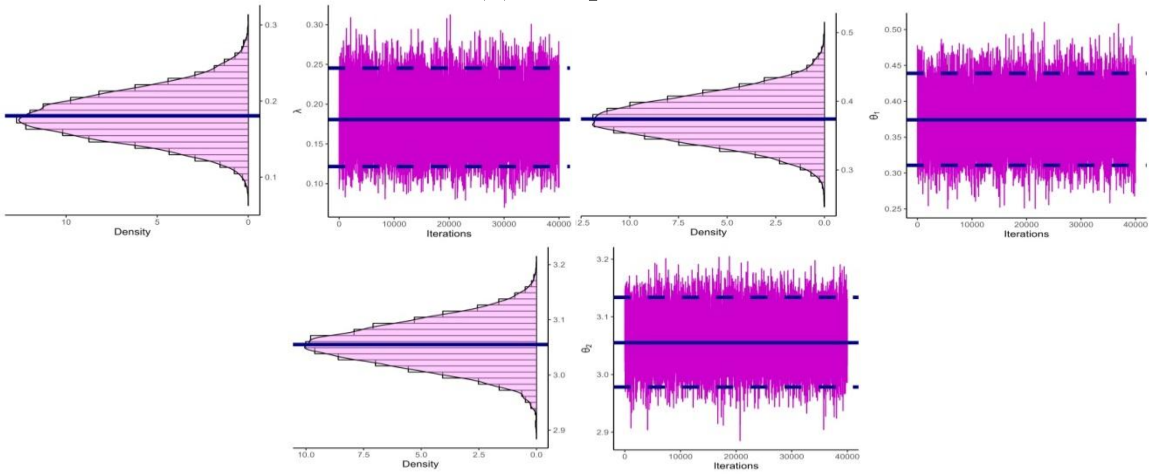
Sample	Par.	Mean	Mode	Q_1	Q_2	Q_3	Std.D	Sk.
A	λ	0.33122	0.24943	0.30006	0.33039	0.36206	0.04576	0.07247
	θ_1	0.24642	0.29239	0.22363	0.24600	0.26881	0.03337	0.09395
	θ_2	2.61337	2.51170	2.57926	2.61306	2.64712	0.04999	0.01993
B	λ	0.18049	0.16915	0.15850	0.17977	0.20191	0.03159	0.13413
	θ_1	0.37409	0.37291	0.35130	0.37348	0.39655	0.03327	0.07356
	θ_2	3.05500	2.99381	3.02782	3.05493	3.08186	0.03972	0.01130
C	λ	0.56053	0.48900	0.53439	0.56020	0.58694	0.03916	0.02535
	θ_1	0.15858	0.17791	0.14264	0.15809	0.17397	0.02348	0.11656
	θ_2	1.88099	1.80891	1.85405	1.88086	1.90775	0.03990	0.00771

Moreover, to display the convergence status of the acquired 40000 Markovian chains, the histogram (with its Gaussian kernel) and trace plots of λ, θ_1 and θ_2 are shown in Figure (2). Specifically, the Bayes point estimate of λ, θ_1 or θ_2 is highlighted by a horizontal solid line, while their 95% BCI bounds are highlighted by horizontal dashed lines. As a result, from Figure (2), it is observed that:

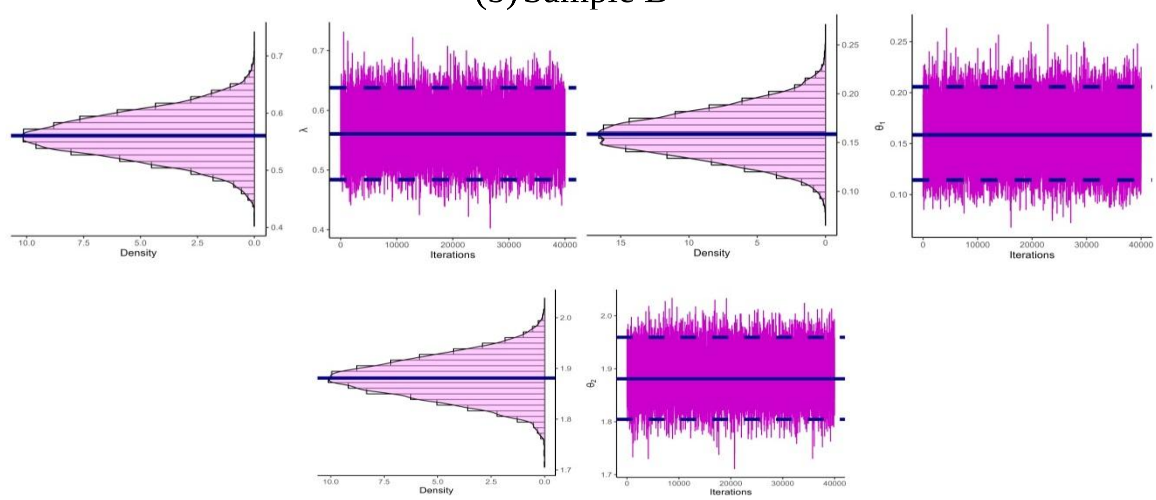
- All acquired estimates by the MCMC algorithm have sufficient convergence.
- The burn-in sample has enough size to eliminate the effect of the starting points.
- The density distribution of λ, θ_1 or θ_2 is almost fairly symmetrical.



(a) Sample A



(b) Sample B



(c) Sample C

Figure 2. The density (left) and trace (right) plots of λ, θ_1, and θ_2 from solar lighting device data.

6.2 Set 2: Carbon Fiber

In this application, to show the utility of the offered estimation methodologies and to verify how our estimates work in practice, a data set consisting of sixty-six observations representing the breaking stress of carbon fibers of 50 mm length (measured in GPa) is examined. Recently, Migdadi et al. (2023), in the context of analyzing k-level SS accelerated life data, provided and discussed this set of data. We shall analyze the simple SS data set with $(n,\tau)=(20,1.81)$, see Table (15).

Table 15. Times of breaking stress of carbon fibers.

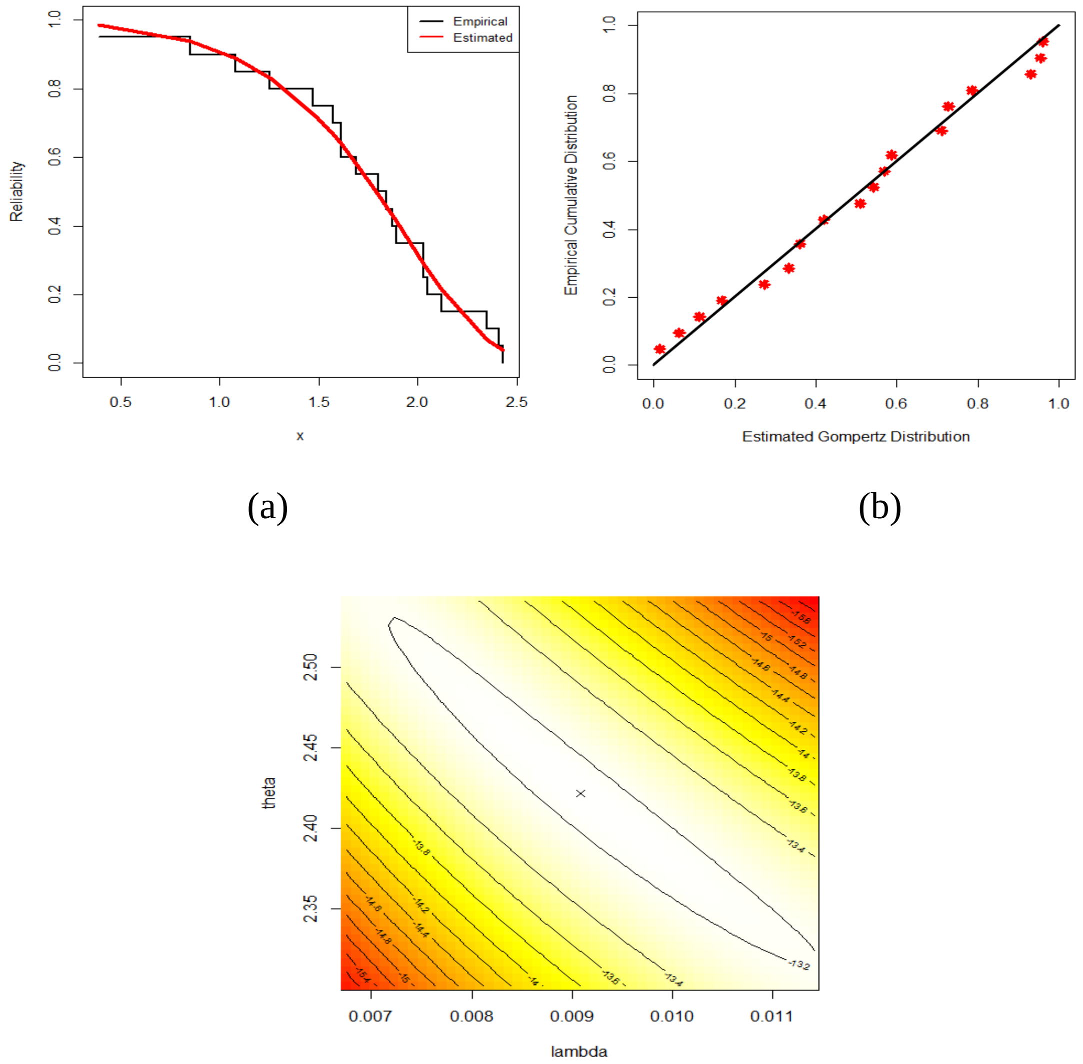
Level 1 $(\tau < 1.81)$
0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80
Level 2 $(\tau > 1.81)$
1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43

The MLEs (with their Std.Ers), 95% ACIs (with their lengths) of λ and θ, and KS(P-value) are obtained in Table (16), the P-value is greater than the significance level of 0.05, hence we cannot reject the null hypothesis, which indicates that the GD fits the carbon fibers data set quite satisfactorily. As we anticipated, from Figure (3), the estimated/empirical reliability lines as well as the PP lines support the same findings reported in Table (16). Additionally, the contour diagram of λ and θ depicted in Figure (3) showed that the offered estimates $\hat{\lambda}=0.0091$ and $\hat{\theta}=2.4217$ of λ and θ, respectively, may exist and are unique.

Table 16. Fit results of the GD from carbon fibers data.

Par.	MLE		95% ACI			KS(P-value)
	Est.	Std.Er	Lower	Upper	Length	
λ	0.0091	0.0084	0.0000	0.0262	0.0262	0.0836(0.999)
θ	2.4217	0.3971	1.6433	3.1998	1.5565	

To examine the offered point and interval estimates of λ, θ_1 and θ_2, various progressively type-II censored samples with a simple SSALT are generated in Table (17). Next, the point estimates (with their Std.Ers) and the interval estimates (with their lengths) of λ, θ_1 and θ_2 are evaluated in Tables (18) and (19). Taking $\mu_i = \alpha_i = 0.001$ for $i = 1, 2, 3$, all Bayes point estimations of λ, θ_1 and θ_2 are assessed based on the SE and LINEX (for $h = (-5, -0.05, +5)$) loss functions.



(c)

Figure 3. The fitted reliability line (a), PP (b), and contour (c) diagrams of the Gompertz lifetime model from carbon fibers data.

Using the proposed M-H steps, for each unknown quantity, we eliminate the first 10000 iterations from the total 50000 MCMC iterations to ignore the influence of the starting points. The results in Tables (18) and (19), in terms of minimum Std.Ers and interval lengths showed that the point and interval estimates of λ, θ_1 and θ_2 obtained via the Bayes approach perform better than other estimates.

Table 17. Three simple SS samples from carbon fibers data when $(n,r)=(20,10)$.

Sample	Scheme	Data
A	$(10,0^9)$	0.39, 1.08, 1.25, 1.47, 1.57, 1.61, 1.80, 1.87, 2.03, 2.12
B	$(5,0^8,5)$	0.39, 0.85, 1.47, 1.57, 1.61, 1.69, 1.84, 1.89, 2.03, 2.03
C	$(1^5,1^5)$	0.39, 1.08, 1.25, 1.57, 1.80, 1.87, 1.89, 2.05, 2.12, 2.35

Table 18. The point estimates (1st Col.) with their Std.Ers (2nd Col.) from carbon fibers data.

Sample	Par.	MLE		SEL		LINEX					
$h \rightarrow$						-5		-0.05		+5	
A	λ	0.0105	0.0168	0.0104	0.0096	0.0104	0.0083	0.0104	0.0086	0.0104	0.0088
	θ_1	2.5778	0.9165	2.5778	0.0099	2.5778	0.0020	2.5778	0.0022	2.5778	0.0025
	θ_2	3.2291	1.4906	3.2291	0.0103	3.2291	0.0041	3.2291	0.0066	3.2291	0.0091
B	λ	0.0176	0.0301	0.0174	0.0229	0.0174	0.0251	0.0174	0.0263	0.0174	0.0276
	θ_1	1.8513	0.9192	1.8512	0.0251	1.8512	0.0068	1.8512	0.0083	1.8512	0.0099
	θ_2	3.4272	1.5646	3.4271	0.0251	3.4271	0.0054	3.4271	0.0070	3.4271	0.0086
C	λ	0.0201	0.0376	0.0200	0.0100	0.0200	0.0054	0.0200	0.0056	0.0200	0.0059
	θ_1	1.6013	0.9909	1.6012	0.0099	1.6012	0.0017	1.6012	0.0020	1.6012	0.0022
	θ_2	2.9201	0.9905	2.9201	0.0103	2.9201	0.0029	2.9201	0.0054	2.9201	0.0079

Table 19. The interval estimates from carbon fibers data.

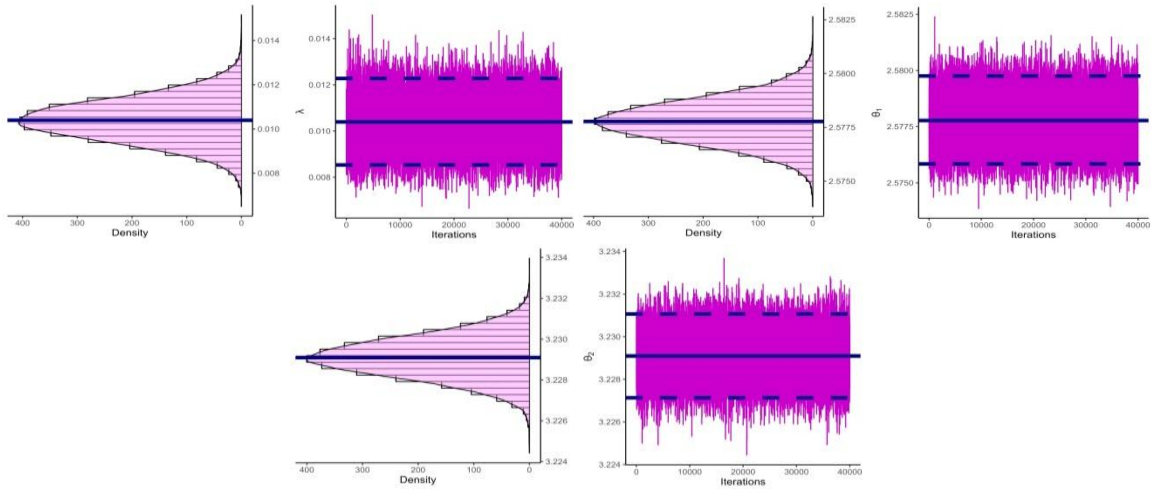
Sample	Par.	95% ACI			95% BCI		
		Lower	Upper	Length	Lower	Upper	Length
A	λ	0.0000	0.0433	0.0433	0.0085	0.0123	0.0037
	θ_1	0.7815	4.3740	3.5925	2.5758	2.5798	0.0039
	θ_2	0.3075	6.1507	5.8432	3.2271	3.2311	0.0039

B	λ	0.0000	0.0767	0.0767	0.0130	0.0219	0.0090
	θ_1	0.0496	3.6530	3.6034	1.8463	1.8561	0.0098
	θ_2	0.3606	6.4938	6.1332	3.4222	3.4320	0.0098
C	λ	0.0000	0.0938	0.0938	0.0181	0.0220	0.0039
	θ_1	0.0000	3.5434	3.5434	1.5993	1.6032	0.0039
	θ_2	0.9780	4.8621	3.8841	2.9181	2.9220	0.0039

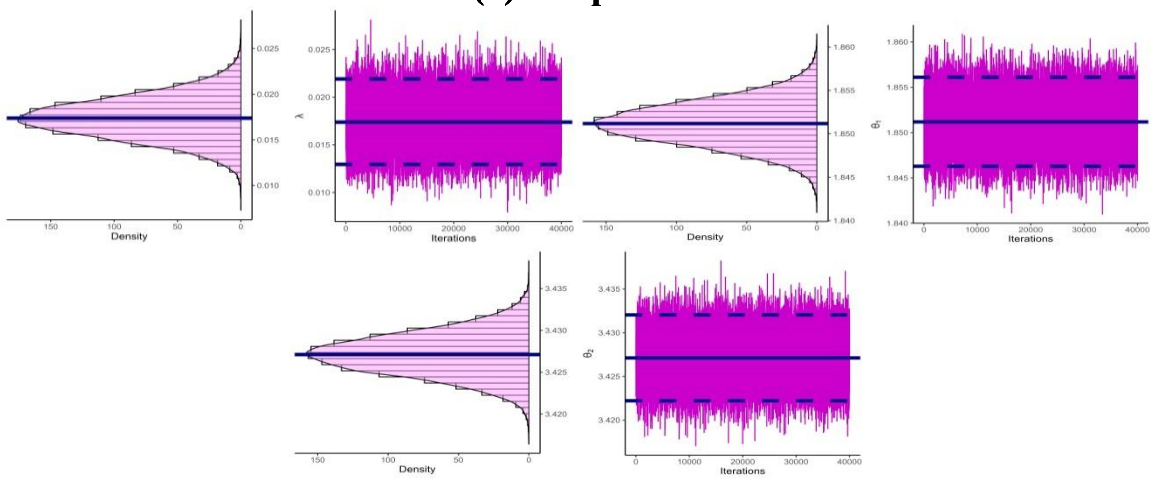
Moreover, in Table (20), several MCMC characteristics of λ,θ_1 and θ_2 based on their staying 40000 iterations are provided. It also supports the same facts reported in Table (18) and shows that the acquired iterations of λ,θ_1 or θ_2 are almost symmetrical. Furthermore, Figure (4) indicates that the MCMC procedure converges very well and that the generated estimates of λ,θ_1 or θ_2 are fairly symmetric. Finally, the proposed inferential methods operate well when applied to real-world data and offer an adequate interpretation of the GD when a sample is created using the recommended censoring plan.

Table 20. Summary of MCMC outputs of λ, θ_1, and θ_2 from carbon fibers data.

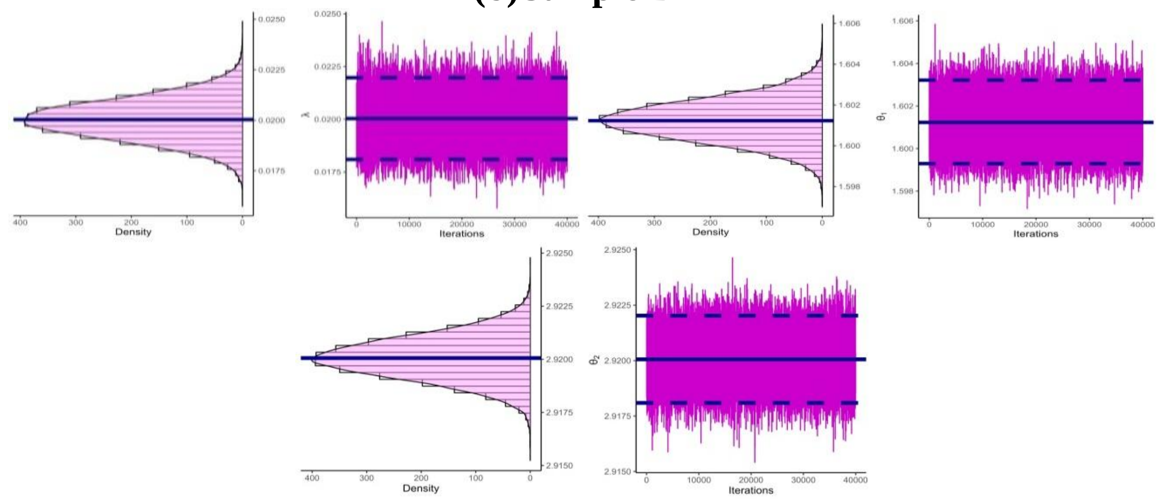
Sample	Par.	Mean	Mode	Q_1	Q_2	Q_3	Std.D	Sk.
A	λ	0.01040	0.00893	0.00975	0.01039	0.01105	0.00096	0.02630
	θ_1	2.57777	2.57585	2.57710	2.57777	2.57845	0.00100	0.03128
	θ_2	3.22910	3.23013	3.22843	3.22909	3.22977	0.00100	-0.00511
B	λ	0.01738	0.01494	0.01585	0.01736	0.01889	0.00228	0.06536
	θ_1	1.85118	1.85069	1.84950	1.85118	1.85287	0.00251	0.00549
	θ_2	3.42711	3.42833	3.42540	3.42712	3.42880	0.00251	0.00832
C	λ	0.02003	0.01804	0.01936	0.02003	0.02070	0.00099	0.01397
	θ_1	1.60123	1.60151	1.60055	1.60123	1.60191	0.00100	0.02790
	θ_2	2.92006	2.91934	2.91938	2.92005	2.92074	0.00100	-0.00329



(a) Sample A



(b) Sample B



(c) Sample C

Figure 4. The density (left) and trace (right) plots of λ, θ_1, and θ_2 from carbon fibers data.

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