Stress-Strength Analysis of Inverse Weibull Model Using Type-II Progressive Hybrid Censoring and Its Application to Light-Emitting Diodes

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Abstract

In this paper, we discuss the estimation of $\delta = P(Y < X)$ based on Type-II progressive hybrid censored samples when X and Y are two independent Inverse Weibull distributions with different scale parameters, but having the same shape parameter. Different methods for estimating δ are applied. The maximum likelihood estimator and the The observed Fisher information matrix is computed and it is used to construct an asymptotic confidence interval for δ. Bayes estimate of δ under the assumptions of independent gamma priors. Markov Chain Monte Carlo (MCMC) technique is used for Bayes computation. Moreover, by using the MCMC method, we achieve the highest posterior density (HPD) credible intervals. Monte Carlo simulations are performed to compare the efficiency of the proposed estimators. One data analysis has been presented for illustrative purposes.

Keywords: Stress-strength model, Inverse Weibull distribution, maximum likelihood estimator, Bayes estimator, Markov Chain Monte Carlo,, Type-II progressively hybrid censoring.

1 Introduction

One significant use of reliability theory is the stress-strength reliability $\delta = P(Y < X)$ model.This model is used in many applications of physics and engineering such as strength failure and system collapse. In electrical and electronic systems δ arise as a measure of system performance. Some authors used δ as a broad measure of the difference between two populations. Dagum [8] utilized δ to quantify inequality between income distributions. (Adimari and Chiogna [1]) utilized it to indicate the evaluation of the area under the receiver operating characteristic (*ROC*) curve for diagnostic tests that produce continuous results. For more details and applications of δ, see Kotz et al. [17].

Statistical inference about the reliability model has gotten a lot of attention in the field of reliability. For $P(Y < X)$,X is the strength of a system which is subjected to stress Y . The system fails when the stress surpasses the strength. Therefore, the stress-strength parameter δ assesses system reliability. In statistical science, estimating δ has been an attention problem for statisticians since 1956 starting with the work of Birnbaum [5]. Since that time, δ has been estimated from both frequentist and Bayesian viewpoints. Recently, some authors have studied the estimation of the stress–strength parameter, such as (Kundu and Gupta [19]), (Babayi and Khorram [3]), Nadar et al. [22], (Nadar and Kizilaslan [21]) and (Kizilaslan and Nadar [14]). Despite extensive research on the stress-strength model in complete sample cases, censored sample studies have received less attention of This parameter. However, in some real instances, for various reasons, Researchers confronted the censored data. Recently, some authors have studied the estimation of the stress–strength parameter, such as (Shoaee and Khorram [26], [27]), Kohansal [15], (Kohansal and Nadarajah[16]), Yadav et al. [29] and Alshenawy et al.[2].

Type-I and Type-II censoring schemes are the two most critical schemes for the study of censored data theory. A mixture of Type-I and Type-II schemes, which has been introduced by Epstein [10], is the hybrid censoring scheme. In the hybrid scheme, during the experiment, the active units cannot be removed. So, Type-I and II progressive hybrid censoring (PHC) schemes have been introduced by Kundu and Joarder [20] and Childs et al. [7]. The Type-I progressive hybrid scheme can be illustrated as follows. Consider one progressive censoring scheme $\{R_1,...,R_r\}$ with n units on the test. The test stopping time $T^* = \min\{X_{(r)}, T\}$, where $T > 0$ and $X_{(1)} \leq \cdots \leq X_{(r)}$ are a fixed time and a progressive censoring sample, respectively. The number of observed samples in the Type-I progressive hybrid scheme may be small. So, the Type-II progressive hybrid (TII-PHC) scheme overcomes the drawback of the Type-I progressive hybrid scheme. The TII-PHC scheme involves the end of experiment at time $T^* = \max\{X_{(r)}, T\}$. It is obvious that if $X_{(r)} \geq T$, then we end the test at time $X_{(r)}$, and $\{X_{(1)}, \ldots, X_{(r)}\}$ is the observed progressive sample. In this case, we confront the progressive scheme. Some authors studied the progressive scheme properties, such as Soliman et al. [28]. For further details on progressively censoring, the book by (Balakrishnan and Aggarwala [4]) may be a suitable reference for readers. If $X_{(r)} < T$, the process will not only follow the prespecified scheme to remove the units after each failure, but continue to observe failures (without any further withdrawals) up to time T.

So, in this censoring, the scheme is $\{R_1,...,R_r,R_{r+1},...,R_D\}$, where $R_r = R_{r+1} = \cdots = R_D = 0$, and the observed sample is

$\{X_{(1)},...,X_{(r)},X_{(r+1)},...,X_{(D)}\}$, this paper obtains some different point and interval estimates of

the reliability parameter, $\delta = P(Y < X)$, when X and Y follow two independent Inverse Weibull distributions (IWD).

The Inverse Weibull distribution(IWD) was introduced by Keller and Kamath [13] with scale and shape parameters as σ and α, respectively, denoted by IW(σ,α) has the probability density function (pdf) and the cumulative distribution function (cdf) respectively is investigated.

$f(x;\sigma,\alpha) = \sigma\alpha x^{-(1+\alpha)} e^{-\sigma x^{-\alpha}}, x > 0, \alpha, \sigma > 0$ (1) $F(x;\sigma,\alpha) = e^{-\sigma x^{-\alpha}}, x > 0, \alpha, \sigma > 0$ (2)

In this paper, we discuss the estimation of δ, when $X \sim IW(\sigma_1,\alpha)$ and $Y \sim IW(\sigma_2,\alpha)$ are independent random variables Based on the TII-PHC scheme. The rest of this paper is organized as follows. We obtain the Stress Strength Parameter in Section II. The MLE of δ in Section III. The asymptotic confidence interval for δ in Section IV. The Bayes estimate of δ by using the MCMC method, We achieve the highest posterior density (HPD) credible intervals in Section V. The simulation study in Section VI. The analysis of real data set is presented in Section VII. Finally the conclusion is given in Section VIII.

2 Stress Strength Parameter

Let $X \sim IW(\sigma_1,\alpha)$ and $Y \sim IW(\sigma_2,\alpha)$ be two independent random variables with the same shape parameter α and $\delta = P(Y < X)$ is the stress-strength reliability model, then:

$$\delta = P(Y < X) = \int_0^\infty \int_0^x \left[\sigma_2 \alpha y^{-(\alpha+1)} e^{-\sigma_2 y^{-\alpha}} dy\right] \sigma_1 \alpha x^{-(\alpha+1)} e^{-\sigma_1 x^{-\alpha}} dx,$$

$$\delta = \frac{\sigma_1}{\sigma_1 + \sigma_2}. \quad \textbf{(3)}$$

3 Maximum Likelihood Estimation of δ

Let $\{X_{(1)}, \ldots, X_{(r)}, X_{(r+1)}, \ldots, X_{(D_1)}\}$ and $\{Y_{(1)}, \ldots, Y_{(k)}, Y_{(k+1)}, \ldots, Y_{(D_2)}\}$ be two TII-PHC sam-
ples with the schemes $\{n,r,T_1,D_1,R_1,\ldots,R_{r-1}\}$ and $\{m,k,T_2,D_2,S_1,\ldots,S_{k-1}\}$, respectively. Under these assumptions, we can write the likelihood function of the unknown parameters as

$$\begin{aligned} L(\sigma_1, \sigma_2, \alpha \mid \underline{x}, \underline{y}) \propto & \prod_{i=1}^{r} f\left(x_{(i)}\right) \left[1 - F\left(x_{(i)}\right)\right]^{R_i} \prod_{i=r+1}^{D_1} f\left(x_{(i)}\right) \left[1 - F(T_1)\right]^{R^*_{D_1}} \\ & \times \prod_{j=1}^{k} f\left(y_{(j)}\right) \left[1 - F\left(y_{(j)}\right)\right]^{S_j} \prod_{j=k+1}^{D_2} f\left(y_{(j)}\right) \left[1 - F(T_2)\right]^{S^*_{D_2}} \end{aligned}$$

where

($n - r -$ Pri=1−1 Ri, if $x(r) \geq T$;, $S_k = \begin{cases} m - k - \sum_{j=1}^{k-1} S_j \\ 0, \end{cases}$ if $y(k) \geq T$;

$R_r = 0$, if $x_{(r)} < T$.if $y_{(k)} < T$. if $x_{(r)} \geq T$;if $y_{(k)} \geq$

$R^*_{D_1} = \begin{cases} 0, \\ n - D_1 - \sum_{i=1}^{r-1} R_i \end{cases}$ T; , if $x_{(r)} < T$., if $y_{(k)} < T$. if $x_{(r)}$ $S^*_{D_2} = \begin{cases} 0, \\ m - D_2 - \sum_{j=1}^{k-1} S_j \end{cases}$

$D_1 = \begin{cases} 0, \\ D_1 \end{cases}$ $\geq T$;if $y_{(k)} \geq T$; , if $x_{(r)} < T$., if $D_2 = \begin{cases} 0, \\ D_2 \end{cases}$ $y_{(k)} < T$.

the likelihood function, based on the observed data, is

$$
\begin{aligned}
L(\sigma_1,\sigma_2,\alpha \mid \underline{x},\underline{y}) \propto & [(\sigma_1\alpha)^r \prod_{i=1}^{r} \left(x_{(i)}\right)^{-(\alpha+1)} e^{-\sigma_1[\sum_{i=1}^{r} x_{(i)}^{-\alpha}+\sum_{i=r+1}^{D_1} x_{(i)}^{-\alpha}]} \\
& \prod_{i=1}^{r}\left(1-e^{-\sigma_1 x_{(i)}^{-\alpha}}\right)^{R_i} (\sigma_1\alpha)^{D_1} \prod_{i=r+1}^{D_1} \left(x_{(i)}\right)^{-(\alpha+1)} \left(1-e^{-\sigma_1 T_1^{-\alpha}}\right)^{R^*_{D_1}}] \\
& \times[(\sigma_2\alpha)^k \prod_{j=1}^{k} \left(y_{(j)}\right)^{-(\alpha+1)} e^{-\sigma_2[\sum_{j=1}^{k} y_{(j)}^{-\alpha}+\sum_{j=k+1}^{D_2} y_{(j)}^{-\alpha}]} \\
& \prod_{j=1}^{k}\left(1-e^{-\sigma_2 y_{(j)}^{-\alpha}}\right)^{S_j} (\sigma_2\alpha)^{D_2} \prod_{j=k+1}^{D_2} \left(y_{(j)}\right)^{-(\alpha+1)} \left(1-e^{-\sigma_2 T_2^{-\alpha}}\right)^{S^*_{D_2}}]
\end{aligned}
\quad \textbf{(4)}
$$

taking the natural logarithm of (4), we obtain

$$
\begin{aligned}
\ln L(\sigma_1,\sigma_2,\alpha \mid \underline{x},\underline{y}) \propto & r\ln(\sigma_1\alpha) + D_1\ln(\sigma_1\alpha) + k\ln(\sigma_2\alpha) + D_2\ln(\sigma_2\alpha) - (\alpha+1) \\
& \left[\sum_{i=1}^{r} \ln\left(x_{(i)}\right) + \sum_{i=r+1}^{D_1} \ln\left(x_{(i)}\right) + \sum_{j=1}^{k} \ln\left(y_{(j)}\right) + \sum_{j=k+1}^{D_2} \ln\left(y_{(j)}\right)\right] \\
& - \sigma_1\left[\sum_{i=1}^{r} x_{(i)}^{-\alpha} + \sum_{i=r+1}^{D_1} x_{(i)}^{-\alpha}\right] + \sum_{i=1}^{r} R_i \ln\left(1-e^{-\sigma_1 x_{(i)}^{-\alpha}}\right) \\
& - \sigma_2\left[\sum_{j=1}^{k} y_{(j)}^{-\alpha} + \sum_{j=k+1}^{D_2} y_{(j)}^{-\alpha}\right] + R^*_{D_1} \ln\left(1-e^{-\sigma_1 T_1^{-\alpha}}\right) \\
& + \sum_{j=1}^{k} S_j \ln\left(1-e^{-\sigma_2 y_{(j)}^{-\alpha}}\right) + S^*_{D_2} \ln\left(1-e^{-\sigma_2 T_2^{-\alpha}}\right)
\end{aligned}
\quad \textbf{(5)}
$$

to derive $\hat{\sigma}_1$,$\hat{\sigma}_2$, and $\hat{\alpha}$, the MLEs of σ_1,σ_2, and α, respectively, we should solve the following equations:

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma_1} =& \frac{r}{\sigma_1} + \frac{D_1}{\sigma_1} - \left[\sum_{i=1}^{r} x_{(i)}^{-\alpha} + \sum_{i=r+1}^{D_1} x_{(i)}^{-\alpha}\right] \\ &+ \sum_{i=1}^{r} R_i \frac{e^{-\sigma_1 x_{(i)}^{-\alpha}} x_{(i)}^{-\alpha}}{1-e^{-\sigma_1 x_{(i)}^{-\alpha}}} + R_{D_1}^{*} \frac{e^{-\sigma_1 T_1^{-\alpha}} T_1^{-\alpha}}{1-e^{-\sigma_1 T_1^{-\alpha}}} = 0\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma_2} =& \frac{k}{\sigma_2} + \frac{D_2}{\sigma_2} - \left[\sum_{j=1}^{k} y_{(j)}^{-\alpha} + \sum_{j=k+1}^{D_2} y_{(j)}^{-\alpha}\right] \\ &+ \sum_{j=1}^{k} S_j \frac{e^{-\sigma_2 y_{(j)}^{-\alpha}} y_{(j)}^{-\alpha}}{1-e^{-\sigma_2 y_{(j)}^{-\alpha}}} + S_{D_2}^{*} \frac{e^{-\sigma_2 T_2^{-\alpha}} T_2^{-\alpha}}{1-e^{-\sigma_2 T_2^{-\alpha}}} = 0\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \alpha} =& \frac{r}{\alpha} + \frac{D_1}{\alpha} + \frac{k}{\alpha} + \frac{D_2}{\alpha} \\ &- \left[\sum_{i=1}^{r} \ln\left(x_{(i)}\right) + \sum_{i=r+1}^{D_1} \ln\left(x_{(i)}\right) + \sum_{j=1}^{k} \ln\left(y_{(j)}\right) + \sum_{j=k+1}^{D_2} \ln\left(y_{(j)}\right)\right] \\ &+ \sigma_1 \left[\sum_{i=1}^{r} x_{(i)}^{-\alpha} \ln\left(x_{(i)}\right) + \sum_{i=r+1}^{D_1} x_{(i)}^{-\alpha} \ln\left(x_{(i)}\right)\right] \\ &+ \sigma_2 \left[\sum_{j=1}^{k} y_{(j)}^{-\alpha} \ln\left(y_{(j)}\right) + \sum_{j=k+1}^{D_2} y_{(j)}^{-\alpha} \ln\left(y_{(j)}\right)\right] \\ &- \sum_{i=1}^{r} \sigma_1 R_i \frac{x_{(i)}^{-\alpha} e^{-\sigma_1 x_{(i)}^{-\alpha}} \ln\left(x_{(i)}\right)}{1-e^{-\sigma_1 x_{(i)}^{-\alpha}}} - \sum_{j=1}^{k} \sigma_2 S_j \frac{y_{(j)}^{-\alpha} e^{-\sigma_2 y_{(j)}^{-\alpha}} \ln\left(y_{(j)}\right)}{1-e^{-\sigma_2 y_{(j)}^{-\alpha}}} \\ &- \sigma_1 R_{D_1}^{*} \frac{T_1^{-\alpha} e^{-\sigma_1 T_1^{-\alpha}} \ln(T_1)}{1-e^{-\sigma_1 T_1^{-\alpha}}} - \sigma_2 S_{D_2}^{*} \frac{T_2^{-\alpha} e^{-\sigma_2 T_2^{-\alpha}} \ln(T_2)}{1-e^{-\sigma_2 T_2^{-\alpha}}} = 0\end{aligned} \tag{8}$$

The MLEs $\hat{\sigma}_1, \hat{\sigma}_2, \hat{\alpha}$ of the model parameters are the solution of non-linear Eqs. (6)-(8) after setting them equal to zero. These equations are very difficult to be solved, so iterative procedures are used as Newton Raphson or conjugate-gradient.

4 Asymptotic Confidence Interval of δ

the asymptotic confidence interval is obtained by deriving the asymptotic distribution of δ. Since δ is a function of the parameters, we first obtain the asymptotic distribution of $\hat{\theta} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\alpha})$. If $I(\theta) = [I_{ij}] = E\left[-\frac{\partial^2 \ln L}{\partial\theta_i \partial\theta_j}\right], i, j = 1, 2, 3$, is the observed Fisher information matrix, then we achieve

the elements of $I(\theta)$ by obtaining the second partial derivatives of function (5) as follows:

$$I(\sigma_1, \sigma_2, \alpha) = -\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \sigma_1^2} & \frac{\partial^2 \ln L}{\partial \sigma_1 \partial \sigma_2} & \frac{\partial^2 \ln L}{\partial \sigma_1 \partial \alpha} \\ \frac{\partial^2 \ln L}{\partial \sigma_2 \partial \sigma_1} & \frac{\partial^2 \ln L}{\partial \sigma_2^2} & \frac{\partial^2 \ln L}{\partial \sigma_2 \partial \alpha} \\ \frac{\partial^2 \ln L}{\partial \alpha \partial \sigma_1} & \frac{\partial^2 \ln L}{\partial \alpha \partial \sigma_2} & \frac{\partial^2 \ln L}{\partial \alpha^2} \end{bmatrix}_{(\sigma_1=\hat{\sigma}_1, \sigma_2=\hat{\sigma}_2, \alpha=\hat{\alpha})},$$

where

$$\frac{\partial^2 \operatorname{Ln} L}{\partial \sigma_1^2} = I_{11} = \frac{-r}{\sigma_1^2} - \frac{D_1}{\sigma_1^2} - \sum_{i=1}^{r} R_i \frac{x_{(i)}^{-2\alpha} e^{-\sigma_1 x_{(i)}^{-\alpha}}}{\left(1 - e^{-\sigma_1 x_{(i)}^{-\alpha}}\right)^2} - R_{D_1}^{*} \frac{T_1^{-2\alpha} e^{-\sigma_1 T_1^{-\alpha}}}{\left(1 - e^{-\sigma_1 T_1^{-\alpha}}\right)^2},$$

$$\frac{\partial^2 \ln L}{\partial \sigma_1 \partial \sigma_2} = I_{12} = 0,$$

$$\frac{\partial^2 \ln L}{\partial \sigma_1 \partial \alpha} = I_{13} = \ln\left(x_{(i)}\right) \left[\sum_{i=1}^{r} x_{(i)}^{-\alpha} + \sum_{i=r+1}^{D_1} x_{(i)}^{-\alpha}\right] + \sum_{i=1}^{r} R_i \frac{x_{(i)}^{-\alpha} \ln\left(x_{(i)}\right) e^{-\sigma_1 x_{(i)}^{-\alpha}} \left[\sigma_1 x_{(i)}^{-\alpha} - 1 + e^{-\sigma_1 x_{(i)}^{-\alpha}}\right]}{\left(1 - e^{-\sigma_1 x_{(i)}^{-\alpha}}\right)^2} + R_{D_1}^{*} \frac{T_1^{-\alpha} \ln(T_1) e^{-\sigma_1 T_1^{-\alpha}} \left[\sigma_1 T_1^{-\alpha} - 1 + e^{-\sigma_1 T_1^{-\alpha}}\right]}{\left(1 - e^{-\sigma_1 T_1^{-\alpha}}\right)^2},$$

$$\frac{\partial^2 \ln L}{\partial \sigma_2^2} = I_{22} = \frac{-k}{\sigma_2^2} - \frac{D_2}{\sigma_2^2} - \sum_{j=1}^{k} S_j \frac{y_{(j)}^{-2\alpha} e^{-\sigma_2 y_{(j)}^{-\alpha}}}{\left(1 - e^{-\sigma_2 y_{(j)}^{-\alpha}}\right)^2} - S_{D_2}^{*} \frac{T_2^{-2\alpha} e^{-\sigma_2 T_2^{-\alpha}}}{\left(1 - e^{-\sigma_2 T_2^{-\alpha}}\right)^2},$$

$$\frac{\partial^2 \ln L}{\partial\sigma_2\partial\alpha} = I_{23} = \ln\left(y_{(j)}\right)\left[\sum_{j=1}^{k} y_{(j)}^{-\alpha} + \sum_{j=k+1}^{D_2} y_{(j)}^{-\alpha}\right]$$
$$+ \sum_{j=1}^{k} s_j \frac{y_{(j)}^{-\alpha}\ln\left(y_{(j)}\right)e^{-\sigma_2 y_{(j)}^{-\alpha}}\left[\sigma_2 y_{(j)}^{-\alpha} - 1 + e^{-\sigma_2 y_{(j)}^{-\alpha}}\right]}{\left(1 - e^{-\sigma_2 y_{(j)}^{-\alpha}}\right)^2}$$
$$+ S_{D_2}^{*}\frac{T_2^{-\alpha}\ln(T_2)e^{-\sigma_2 T_2^{-\alpha}}\left[\sigma_2 T_2^{-\alpha} - 1 + e^{-\sigma_2 T_2^{-\alpha}}\right]}{\left(1 - e^{-\sigma_2 T_2^{-\alpha}}\right)^2},$$

$$\frac{\partial^2 \ln L}{\partial\alpha^2} = I_{33} = \frac{-r}{\alpha^2} - \left[\frac{D_1 + D_2}{\alpha^2}\right] - \frac{k}{\alpha^2} - \sigma_1 \ln^2\left(x_{(i)}\right)\left[\sum_{i=1}^{r} x_{(i)}^{-\alpha} + \sum_{i=r+1}^{D_1} x_{(i)}^{-\alpha}\right]$$
$$- \sigma_2 \ln^2\left(y_{(j)}\right)\left[\sum_{j=1}^{k} y_{(j)}^{-\alpha} + \sum_{j=k+1}^{D_2} y_{(j)}^{-\alpha}\right]$$
$$- \sum_{i=1}^{r} R_i\sigma_1 \ln\left(x_{(i)}\right)\frac{x_{(i)}^{-\alpha}\ln\left(x_{(i)}\right)e^{-\sigma_1 x_{(i)}^{-\alpha}}\left[\sigma_1 x_{(i)}^{-\alpha} - 1 + e^{-\sigma_1 x_{(i)}^{-\alpha}}\right]}{\left(1 - e^{-\sigma_1 x_{(i)}^{-\alpha}}\right)^2}$$
$$- R_{D_1}^{*}\ln(T_1)\frac{\sigma_1 T_1^{-\alpha}\ln(T_1)e^{-\sigma_1 T_1^{-\alpha}}\left[\sigma_1 T_1^{-\alpha} - 1 + e^{-\sigma_1 T_1^{-\alpha}}\right]}{\left(1 - e^{-\sigma_1 T_1^{-\alpha}}\right)^2}$$
$$- \sum_{j=1}^{k} S_j\sigma_2 \ln\left(y_{(j)}\right)\frac{y_{(j)}^{-\alpha}\ln\left(y_{(j)}\right)e^{-\sigma_2 y_{(j)}^{-\alpha}}\left[\sigma_2 y_{(j)}^{-\alpha} - 1 + e^{-\sigma_2 y_{(j)}^{-\alpha}}\right]}{\left(1 - e^{-\sigma_2 y_{(j)}^{-\alpha}}\right)^2}$$
$$- S_{D_2}^{*}\ln(T_2)\frac{\sigma_2 T_2^{-\alpha}\ln(T_2)e^{-\sigma_2 T_2^{-\alpha}}\left[\sigma_2 T_2^{-\alpha} - 1 + e^{-\sigma_2 T_2^{-\alpha}}\right]}{\left(1 - e^{-\sigma_2 T_2^{-\alpha}}\right)^2}.$$

Let $\hat{\sigma}_1, \hat{\sigma}_2$ and $\hat{\alpha}$ be the MLEs of σ_1, σ_2, and α, respectively. So

$$\left[(\hat{\sigma}_1 - \sigma_1), (\hat{\sigma}_2 - \sigma_2), (\hat{\alpha} - \alpha)\right]^T \to N_3\left(0, \mathbf{I}^{-1}(\sigma_1, \sigma_2, \alpha)\right),$$

where $\mathbf{I}(\sigma_1,\sigma_2,\alpha)$ and $\mathbf{I}^{-1}(\sigma_1,\sigma_2,\alpha)$ are symmetric matrices and

$$\mathbf{I}(\sigma_1,\sigma_2,\alpha) = \begin{pmatrix} I_{11} & 0 & I_{13} \\ & I_{22} & I_{23} \\ & & I_{33} \end{pmatrix}, \quad \mathbf{I}^{-1}(\sigma_1,\sigma_2,\alpha) = \frac{1}{|\mathbf{I}(\sigma_1,\sigma_2,\alpha)|}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ & & u_{33} \end{pmatrix}$$

in which

$$|\mathbf{I}(\sigma_1,\sigma_2,\alpha)| = I_{11}I_{22}I_{33} - I_{11}I_{23}^2 - I_{13}^2 I_{22},$$

$$u_{11} = I_{22}I_{33} - I_{23}^2,\quad u_{12} = I_{13}I_{23},\quad u_{13} = -I_{13}I_{22}$$

$$u_{22} = I_{11}I_{33} - I_{13}^2,\quad u_{23} = -I_{11}I_{23},\quad u_{33} = I_{11}I_{22}$$

Let $\hat{\delta}^{MLE}$ be the MLE of δ, we have

$$\left(\hat{\delta}^{MLE} - \delta\right) \to N(0, B) \text{ in distribution}$$

Now, the variance of $\hat{\delta}$, denoted by B, can be obtained use the delta method.

Therefore, $B = \mathbf{u}^T \mathbf{I}^{-1}(\sigma_1,\sigma_2,\alpha)\mathbf{u}$, where

$$B = \frac{1}{|\mathbf{I}(\sigma_1,\sigma_2,\alpha)|}\left[\left(\frac{\partial\delta}{\partial\sigma_1}\right)^2 u_{11} + \left(\frac{\partial\delta}{\partial\sigma_2}\right)^2 u_{22} + 2\left(\frac{\partial\delta}{\partial\sigma_1}\right)\left(\frac{\partial\delta}{\partial\sigma_2}\right) u_{12}\right] \qquad \textbf{(9)}$$

with

$$u = \begin{bmatrix} \frac{\partial \delta}{\partial \sigma_1} \\ \frac{\partial \delta}{\partial \sigma_2} \\ \frac{\partial \delta}{\partial \alpha} \end{bmatrix}^T = \begin{bmatrix} \frac{\partial \delta}{\partial \sigma_1} \\ \frac{\partial \delta}{\partial \sigma_2} \\ 0 \end{bmatrix}^T,$$

in which

$$\frac{\partial \delta}{\partial \sigma_1} = \frac{\sigma_2}{(\sigma_1 + \sigma_2)^2}, \quad \frac{\partial \delta}{\partial \sigma_2} = -\frac{\sigma_1}{(\sigma_1 + \sigma_2)^2} \tag{10}$$

we obtain the asymptotic confidence interval of δ. We noted that using the MLEs of σ_1, σ_2, and α, the value of B should be estimated. Hence, a $100(1 - \gamma)\%$ asymptotic confidence interval of δ can be constructed by

$$\left(\hat{\delta}_{\text{MLE}} - z_{1-\frac{\gamma}{2}}\sqrt{\hat{B}},\ \hat{\delta}_{\text{MLE}} + z_{1-\frac{\gamma}{2}}\sqrt{\hat{B}}\right)$$

where z_γ is 100γ-th percentile of $N(0,1)$.

5 Bayes estimation of δ

We consider the Bayes estimation of δ under the assumption that the shape parameter α and scale parameters σ_1 and σ_2 are random variables. It is assumed the σ_1, σ_2 and α have independent gamma priors with PDFs:

$$\pi(\sigma_1) = \frac{b_1^{a_1}}{\Gamma(a_1)}\sigma_1^{a_1-1}e^{-b_1\sigma_1}, \qquad \sigma_1 > 0, a_1 > 0, b_1 > 0$$

$$\pi(\sigma_2) = \frac{b_2^{a_2}}{\Gamma(a_2)}\sigma_2^{a_2-1}e^{-b_2\sigma_2}, \qquad \sigma_2 > 0, a_2 > 0, b_2 > 0,$$

$$\pi(\alpha) = \frac{b_3^{a_3}}{\Gamma(a_3)}\alpha^{a_3-1}e^{-b_3\alpha}, \qquad \alpha > 0, a_3 > 0, b_3 > 0,$$

where a_1,b_1,a_2,b_2,a_3 and b_3 are the hyper-parameters. Then, the joint prior density of σ_1,σ_2 and α can be written as

$$\pi(\sigma_1,\sigma_2,\alpha) \propto \sigma_1^{a_1-1}\sigma_2^{a_2-1}\alpha^{a_3-1}e^{-(b_1\sigma_1+b_2\sigma_2+b_3\alpha)}, \qquad a_1,a_2,a_3,b_1,b_2,b_3 > 0 \quad \textbf{(11)}$$

Combining equation (4) with equation (11) then

$$\begin{aligned} L(\sigma_1,\sigma_2,\alpha \mid \underline{x},\underline{y})\pi(\sigma_1,\sigma_2,\alpha) \propto\ & \sigma_1^{r+D_1+a_1-1}\sigma_2^{k+D_2+a_2-1} \\ & \alpha^{r+D_1+k+D_2+a_3-1}e^{-b_1\sigma_1}e^{-b_2\sigma_2}e^{-b_3\alpha} \\ & \prod_{i=1}^{r}\left(x_{(i)}\right)^{-(\alpha+1)}e^{-\sigma_1[\sum_{i=1}^{r}x_{(i)}^{-\alpha}+\sum_{i=r+1}^{D_1}x_{(i)}^{-\alpha}]} \\ & \prod_{i=1}^{r}\left(1-e^{-\sigma_1 x_{(i)}^{-\alpha}}\right)^{R_i}\prod_{i=r+1}^{D_1}\left(x_{(i)}\right)^{-(\alpha+1)}\left(1-e^{-\sigma_1 T_1^{-\alpha}}\right)^{R_{D_1}^*} \qquad (12) \\ & \prod_{j=1}^{k}\left(y_{(j)}\right)^{-(\alpha+1)}e^{-\sigma_2[\sum_{j=1}^{k}y_{(j)}^{-\alpha}+\sum_{j=k+1}^{D_2}y_{(j)}^{-\alpha}]} \\ & \prod_{j=1}^{k}\left(1-e^{-\sigma_2 y_{(j)}^{-\alpha}}\right)^{S_j}\prod_{j=k+1}^{D_2}\left(y_{(j)}\right)^{-(\alpha+1)}\left(1-e^{-\sigma_2 T_2^{-\alpha}}\right)^{S_{D_2}^*} \end{aligned}$$

The joint posterior density of σ_1,σ_2 and α can be written as

$$\pi(\sigma_1,\sigma_2,\alpha \mid \underline{x},\underline{y}) = \frac{L(\sigma_1,\sigma_2,\alpha \mid \underline{x},\underline{y})\pi_1(\sigma_1)\pi_2(\sigma_2)\pi_3(\alpha)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\sigma_1,\sigma_2,\alpha \mid \underline{x},\underline{y})\pi_1(\sigma_1)\pi_2(\sigma_2)\pi_3(\alpha)d\sigma_1 d\sigma_2 d\alpha} \quad \textbf{(13)}$$

The joint posterior density of the unknown parameters given in (13) is complicated and no closed form estimates appear to be possible. We, therefore, consider MCMC techniques namely, Gibbs sampler and Metropolis-Hastings (M-H) algorithm to obtain the sample based Bayes estimator of the stress-strength reliability δ and to construct the corresponding HPD credible interval. The full posterior conditional distributions for σ_1, σ_2 and α respectively, are given by

$$\pi_1(\sigma_1 \mid \sigma_2, \alpha, \underline{x}, \underline{y}) \propto \sigma_1^{r+D_1+a_1-1} e^{-b_1\sigma_1} \prod_{i=1}^{r} \left(x_{(i)}\right)^{-(\alpha+1)} e^{-\sigma_1 x_{(i)}^{-\alpha}} \left(1-e^{-\sigma_1 x_{(i)}^{-\alpha}}\right)^{R_i} \prod_{i=r+1}^{D_1} \left(x_{(i)}\right)^{-(\alpha+1)} e^{-\sigma_1 x_{(i)}^{-\alpha}} \left(1-e^{-\sigma_1 T_1^{-\alpha}}\right)^{R^*_{D_1}} \quad \textbf{(14)}$$

$$\pi_2(\sigma_2 \mid \sigma_1, \alpha, \underline{x}, \underline{y}) \propto \sigma_2^{k+D_2+a_2-1} e^{-b_2\sigma_2} \prod_{j=1}^{k} \left(y_{(j)}\right)^{-(\alpha+1)} e^{-\sigma_2 y_{(j)}^{-\alpha}} \left(1-e^{-\sigma_2 y_{(j)}^{-\alpha}}\right)^{S_j} \prod_{j=k+1}^{D_2} \left(y_{(j)}\right)^{-(\alpha+1)} e^{-\sigma_2 y_{(j)}^{-\alpha}} \left(1-e^{-\sigma_2 T_2^{-\alpha}}\right)^{S^*_{D_2}} \quad \textbf{(15)}$$

$$\pi_3(\alpha \mid \sigma_1, \sigma_2, \underline{x}, \underline{y}) \propto \alpha^{r+D_1+k+D_2+a_3-1} e^{-b_3\alpha} \prod_{i=1}^{r} \left(x_{(i)}\right)^{-(\alpha+1)} e^{-\sigma_1 x_{(i)}^{-\alpha}} \left(1-e^{-\sigma_1 x_{(i)}^{-\alpha}}\right)^{R_i} \prod_{i=r+1}^{D_1} \left(x_{(i)}\right)^{-(\alpha+1)} e^{-\sigma_1 x_{(i)}^{-\alpha}} \left(1-e^{-\sigma_1 T_1^{-\alpha}}\right)^{R^*_{D_1}} \prod_{j=1}^{k} \left(y_{(j)}\right)^{-(\alpha+1)} e^{-\sigma_2 y_{(j)}^{-\alpha}} \left(1-e^{-\sigma_2 y_{(j)}^{-\alpha}}\right)^{S_j} \prod_{j=k+1}^{D_2} \left(y_{(j)}\right)^{-(\alpha+1)} e^{-\sigma_2 y_{(j)}^{-\alpha}} \left(1-e^{-\sigma_2 T_2^{-\alpha}}\right)^{S^*_{D_2}} \quad \textbf{(16)}$$

We use following hybrid algorithm to generate samples from the full conditional posterior densities:

Step 1: Set an initial values $\left(\sigma_1^{(0)}, \sigma_2^{(0)}, \alpha^{(0)}\right)$.

Step 2: Let $t = 1$.

Step 3: Generate $\sigma_1^{(t)}$ from $\pi_1(\sigma_1^{(t)} \mid \sigma_2^{(t-1)}, \alpha^{(t-1)}, \underline{x}, \underline{y})$ in (14) using M-H algorithm withnormal proposal distribution.

Step 4: Generate$\sigma_2^{(t)}$ from$\pi_2(\sigma_2^{(t)} \mid \sigma_1^{(t)}, \alpha^{(t-1)}, \underline{x}, \underline{y})$ in (15) using M-H algorithm withnormal proposal distribution.

Step 5: Generate $\alpha^{(t)}$ from$\pi_3(\alpha^{(t)} \mid \sigma_1^{(t)}, \sigma_2^{(t)}, \underline{x}, \underline{y})$ in (16) using M-H algorithm withnormal proposal distribution.

Step 6: Compute $\delta^{(t)}$ from Equation (3).

Step 7: Set $t = t + 1$.

Step 8: Repeat steps 3-7, N times.

Using this algorithm, Bayesian estimation of δ, under the squared error loss function, is derived as

$$\widehat{\delta}^{\mathrm{MC}} = \frac{1}{N-M} \sum_{t=M+1}^{N} \delta_t \tag{17}$$

where M is the burn-in period. Moreover, a $100(1-\gamma)\%$ HPD credible interval of δ can be constructed by applying the method accomplished by Chen and Shao [6].

6 Monte Carlo Simulations

This section evaluates and compares the offered theoretical results for point and interval estimators about σ_1, σ_2, α, and δ based on a series of extensive Monte Carlo simulations.

6.1 Simulation designs

From the proposed IW model when $(\sigma_1,\sigma_2,\alpha) = (1.5,2.5,5)$, we simulate 1,000 TII-PHC samples based on various choices of n,m(complete sample sizes), r,k(effective sample sizes), T_i, $i = 1,2$,(threshold times), and **R**,**S**(progressive censoring patterns). Here, the plausible value of δ from the proposed populations is taken as 0.375.

Specifically, using T_1(=2.5,9.5) and T_2(=5.5,7.5), the values of r (or k) are utilized as a failure percentage (FP), such as FP$[r] = \frac{r}{n} \times 100\%$ = 50 and 80% (as an example). Without loss of generality, in Table 1, different comparison setups of $IW(\sigma_1,\alpha)$ and $IW(\sigma_2,\alpha)$ populations are provided. To distinguish, in Table 1, the progressive design (5*4,0*16) (as an example) means that five survival items will be drawn at each stage of the first four stages, and the experimenter will stop the removals for the remaining sixteen stages.

Table 1: Comparison scenarios used in Monte Carlo simulations.

Test	n(FP[r]%)	m(FP[k]%)	{**R**,**S**}
A	40([50%])	30([50%])	PC[1]:{(5*4,0*16),(5*3,0*12)} PC[2]:{(0*8,5*4,0*8),(0*6,5*3,0*6)} PC[3]:{(0*16,5*4),(0*12,5*3)}
B	40([80%])	30([80%])	PC[1]:{(2*4,0*28),(2*3,0*21)} PC[2]:{(0*14,2*4,0*14),(0*11,2*3,0*10)} PC[3]:{(0*16,5*4),(0*12,5*3)}
C	60([50%])	80([50%])	PC[1]:{(5*6,0*24),(5*8,0*32)} PC[2]:{(0*12,5*6,0*12),(0*16,5*8,0*16)} PC[3]:{(0*24,5*6),(0*32,5*8)}
D	60([80%])	80([80%])	PC[1]:{(2*6,0*42),(2*8,0*56)} PC[2]:{(0*21,2*6,0*21),(0*28,2*8,0*28)} PC[3]:{(0*42,2*6),(0*56,2*8)}

Not, to collect a TII-PHC sample of sizes r and k from the proposed lifetime populations $IW(\sigma_1,\alpha)$ and $IW(\sigma_2,\alpha)$, do the following procedure:

Step 1: Set the actual values σ_1 and α in $IW(\sigma_1,\alpha)$ population.

Step 2: Simulate a traditional progressive Type-II censored sample as:

(a) Obtain ψ *independent* items (say $\psi_1,\psi_2,\ldots,\psi_r$) from uniform $U(0,1)$ distribution.

(b) Set $\Omega_i = \psi_i^{\left(i+\sum_{j=r-i+1}^{r} R_j\right)^{-1}}$, for $i = 1,2,\ldots,r$.

(c) Set $U_i = 1 - \Omega_r\Omega_{r-1}\cdots\Omega_{r-i+1}$ for $i = 1,2,...,r$.

(d) Collect a desired progressive Type-II sample ($X_{(i)}$, $i = 1,2,...,r$) from $IW(\sigma_1,\alpha)$ by setting:

$$X_{(i)} = \left(-\sigma_1^{-1}\log(u_i)\right)^{-\frac{1}{\alpha}},\ i = 1,2,\ldots,r.$$

Step 3: Obtain D_1 at T_1.

Step 4. Determine the TII-PHC data type as:

(a) Case-1: If $T_1 < X_{(r)}$, stop the test at $X_{(r)}$.

(b) Case-2: If $X_{(r)} < T_1$, set $R_i = 0$, $i = r,r+1,...,D_1$, then stop the test at T_1.

Step 5: Redo Steps 2–4 for $IW(\sigma_2,\alpha)$ population.

Once the 1,000 TII-PHC samples are collected, using the package 'maxLik' (by Henningsen and Toomet [11]), the MLEs and 95% ACIs of σ_1, σ_2, α, and δ are evaluated. To carry out the proposed Bayes' estimation, following the idea of prior mean and prior variance of hyper-parameters, which was introduced by Kundu [18], we utilized two informative sets of $(a_1,a_2,a_3,b_1,b_2,b_3)$ called:

- Prior-1:$(a_1,a_2,a_3) = (7.5,12.5,25)$ and $b_i = 5$, $i = 1,2,3$;
- Prior-2:$(a_1,a_2,a_3) = (15,25,50)$ and $b_i = 10$, $i = 1,2,3$.

To generate MCMC samples of σ_1, σ_2, or α, using the 'coda' package (by Plummer et al. [25]) 12,000 MCMC samples are generated, and the first 2,000 variates are eliminated as burn-in. Then, using the remaining 10,000 MCMC samples, the computations of the proposed Bayes point and interval estimates of σ_1, σ_2, α, or δ are obtained. All programming language softwares presented in this study were recently recommended by Elshahhat et al. [9] and developed by R 4.2.2.

Of course, for each setup, we compute the average estimate (AvE) of δ (as an example) as follows:

$$\mathrm{AvE}(\breve{\delta}) = \frac{1}{1000}\sum_{i=1}^{1000} \breve{\delta}^{(i)},$$

where $\breve{\delta}^{(i)}$ represents the acquired estimate of δ at ith sample.

The comparison between the proposed point estimators of δ is made based on two different criteria, namely:

(a) Root Mean Squared-Error (RMSE):

$$\mathrm{RMSE}(\breve{\delta}) = \sqrt{\frac{1}{1000}\sum_{i=1}^{1000}\left(\breve{\delta}^{(i)} - \delta\right)^2}$$

(b) Mean Relative Absolute Bias (MRAB):

$$\mathrm{MRAB}(\breve{\delta}) = \frac{1}{1000}\sum_{i=1}^{1000} \delta^{-1}\left|\breve{\delta}^{(i)} - \delta\right|.$$

On the other hand, the comparison between the proposed interval estimators of δ is made based on two accuracy criteria, namely:

(a) Average Interval Length (AIL):

$$\text{ACL}^{95\%}(\delta) = \frac{1}{1000}\sum_{i=1}^{1000}\left(\mathcal{U}_{\breve{\delta}(i)} - \mathcal{L}_{\breve{\delta}(i)}\right).$$

(b) Coverage Probability (CP):

$$\text{CP}^{95\%}(\delta) = \frac{1}{1000}\sum_{i=1}^{1000}\mathbf{1}^{\star}_{\left(\mathcal{L}_{\breve{\delta}(i)};\mathcal{U}_{\breve{\delta}(i)}\right)}(\delta),$$

where $\mathbf{1}^{\star}(\cdot)$ is the indicator function and (L(·),U(·)) denotes the estimated two interval bounds of δ.

In a similar way, the AvEs, RMSEs, MRABs, AILs, and CPs of σ_1, σ_2, or α can be easily obtained.

6.2 Simulation outputs

All simulation findings are listed in Tables 2-9. To distinguish, in Tables 2-5, the AvEs, RMSEs, and MRABs are listed in the first, second, and third columns, respectively. Furthermore, in Tables 6-9, the AILs and CPs are listed in the first and second columns, respectively.

Now, from Tables 2-9, in terms of the lowest levels of RMSEs, MRABs, and AILs as well as the largest level of CPs, we report the following observations:

- All offered inferential results of σ_1, σ_2, α, or δ behave satisfactorily.
- As r(or k) increases, the associated accuracy of all acquired estimates behaves well. A similar note is also reached when FP[r]%(or FP[k]%) increases.
- As T_i, i = 1,2, increase, it is noted that the simulated RMSE, MRAB, and AIL values for both maximum likelihood and Bayesian estimation results

of σ_1, σ_2, α, and δ increase. Furthermore, the opposite comment is observed when the comparison is made in terms of their CP values.

- Comparing the proposed point estimations, it is clear that the estimates of σ_1, σ_2, α, or δ created from the Bayes method behave better compared to those created from the maximum likelihood method. This result is due to the fact that the Bayesian estimation results involved more priority information on the unknown parameters than the classical estimates.
- Comparing the proposed interval estimations, it is clear that:
  - The results of σ_1, σ_2, α, or δ created by the BCI (or HPD) method outperformed those created by the ACI-NA (or ACI-NL) method. This is an expected result given the fact that the BCI (or HPD) results included more precision information from the proposed joint gamma priors.
  - The results of σ_1, σ_2, and α created by the ACI-NA approach are the best compared to those obtained based on the ACI-NL approach.
  - The results of δ created by the ACI-NL approach are the best compared to those obtained based on the ACI-NA approach.
- Comparing the proposed priors 1 and 2, it is shown that estimates made using the latter are superior to those developed using the former. This is an expected outcome because the variance of previous 2 is lower than that of prior 1.
- Comparing the proposed PC[i], $i = 1,2,3$, we have noted that:
  - All point and interval estimates σ_1, σ_2, or δ behave better based on PC[1] (when remaining live elements are removed during the early stages) than others.

- All point and interval estimates α behave better based on PC[3] (when remaining live elements are removed during the last stages) than others.

- As a recommendation, when the sample is collected from the proposed TII-PHC strategy, the Metropolis-Hastings algorithm is advised to estimate the unknown parameters when the stress and strength follow two independent inverse Weibull populations.

Table 2: Point estimation results of σ_1.

Test	Scheme	MLE			MCMC					
					Prior-1			Prior-2		
					$(T_1,T_2) =$ $(2.5,5.5)$					
					1.4721 1.1627 1.2132 1.0703 1.0151 0.9895 1.3439 1.0562 1.1449 1.1342 1.0955 1.0693 $_2) = (9.5,$					
A	PC[1]	1.4315	0.7196	0.3961	1.5572	0.6988	0.3759	1.7182	0.6036	0.3408
	PC[2]	0.9139	0.7660	0.4330	1.1166	0.7435	0.4142	1.5495	0.6450	0.3896
	PC[3]	1.0266	0.7436	0.4132	1.1689	0.7309	0.4029	1.5045	0.6249	0.3787
B	PC[1]	1.0355	0.6541	0.3676	1.1157	0.6242	0.3313	1.2928	0.5620	0.2999
	PC[2]	0.9058	0.6739	0.3855	0.9948	0.6262	0.3403	1.2196	0.5833	0.3005
	PC[3]	0.8637	0.6865	0.3908	0.9680	0.6816	0.3538	1.2176	0.6029	0.3208
C	PC[1]	1.3326	0.5632	0.3097	1.4129	0.5411	0.3006	1.3613	0.4710	0.1545
	PC[2]	0.9487	0.5949	0.3396	1.0464	0.5793	0.3090	1.1645	0.5137	0.2987
	PC[3]	1.0607	0.5897	0.3156	1.1372	0.5816	0.3073	1.1906	0.4919	0.2317
D	PC[1]	0.9906	0.4358	0.2250	1.1564	0.3607	0.1571	1.0677	0.2325	0.1244
	PC[2]	0.9217	0.5140	0.2434	1.0918	0.3919	0.1974	1.0249	0.2778	0.1355
	PC[3]	0.8801	0.5610	0.2929	1.0624	0.4971	0.2772	1.0114	0.4507	0.1510

Table 3: Point estimation results of σ2.

Test	Scheme	MLE			MCMC					
					Prior-1			Prior-2		
$(T_1,T_2) = (2.5,5.5)$										
					2.1582					
					1.9589					
					1.9204					
					2.1106					
					2.1006					
					2.0992					
					1.7388					
					1.6212					
					1.5436					
					1.4340					
					1.3880					
					1.3505					
					$_2) = (9.5,$					
A	PC[1]	1.6896	1.1507	0.4560	2.1469	1.0594	0.4182	2.4450	0.6597	0.2300
	PC[2]	1.6100	1.1559	0.4611	1.9354	1.1256	0.4447	2.3191	0.7265	0.2529
	PC[3]	1.4951	1.2006	0.4789	1.8858	1.1712	0.4627	2.3372	0.7297	0.2572
B	PC[1]	1.4329	1.0764	0.4275	2.1395	0.7665	0.2977	2.3985	0.4077	0.1380
	PC[2]	1.4035	1.1067	0.4386	2.0974	0.9135	0.3551	2.3967	0.5239	0.1855
	PC[3]	1.3599	1.1204	0.4471	2.0962	0.9911	0.3849	2.3976	0.5338	0.1882
C	PC[1]	1.6360	0.9556	0.3795	1.7560	0.5697	0.2023	2.1551	0.2759	0.0670
	PC[2]	1.5511	1.0299	0.4020	1.6127	0.6362	0.2279	2.0362	0.2838	0.0685
	PC[3]	1.4311	1.0733	0.4268	1.5385	0.6835	0.2460	2.0297	0.2846	0.0736
D	PC[1]	1.3822	0.8138	0.3242	1.4544	0.4752	0.1701	1.9263	0.1752	0.0386
	PC[2]	1.3472	0.8664	0.3456	1.3883	0.5268	0.1902	1.8576	0.2470	0.0658
	PC[3]	1.3027	0.9129	0.3560	1.3432	0.5544	0.1971	1.8688	0.2601	0.0661

Table 4: Point estimation results of **α**.

Test	Scheme	MLE			MCMC					
					Prior-1			Prior-2		
					(T_1,T_2) = (2.5,5.5)					
					4.8781					
					4.7298					
					3.4038					
					4.9918					
					4.9371					
					4.6478					
					4.6674					
					4.4193					
					3.4402					
					4.2416					
					4.1481					
					3.9715					
					2) = (9.5,					
A	PC[1]	4.5934	3.5755	0.7151	5.1628	2.4316	0.4682	5.2157	0.9748	0.2870
	PC[2]	4.2236	2.9043	0.5807	5.0188	2.1449	0.4162	5.1777	0.8949	0.2819
	PC[3]	1.4246	1.9861	0.3963	2.6589	1.3346	0.2482	4.1700	0.6064	0.1978
B	PC[1]	4.2357	1.6581	0.3305	5.1789	0.9399	0.1944	5.2930	0.5652	0.1612
	PC[2]	4.0858	1.2195	0.2359	5.1167	0.9225	0.1898	5.2756	0.5299	0.1576
	PC[3]	3.0186	1.1984	0.2349	4.2458	0.8563	0.1891	4.8356	0.5162	0.1410
C	PC[1]	4.1771	1.0611	0.2063	4.7088	0.6717	0.1851	5.1590	0.5151	0.1344
	PC[2]	3.8205	0.9868	0.1893	4.4784	0.5386	0.1821	5.0698	0.5079	0.1271
	PC[3]	2.0963	0.8943	0.1786	2.9192	0.5384	0.1652	4.1906	0.5072	0.1216
D	PC[1]	3.9684	0.8907	0.1723	4.2956	0.5190	0.1580	4.9908	0.4624	0.1179
	PC[2]	3.8255	0.8581	0.1638	4.1942	0.5021	0.1544	4.9336	0.4396	0.0973
	PC[3]	3.3477	0.7997	0.1571	3.7588	0.4780	0.1474	4.6813	0.3614	0.0923

Table 5: Point estimation results of δ.

0.4055
0.3711
0.3777
0.3334
0.3213
0.3154
0.4325
0.3974
0.4070
0.4356
0.4341
0.4343
$_2) = (9.5,$

Test	Scheme	MLE			MCMC					
					Prior-1			Prior-2		
					$(T_1,T_2) = (2.5,5.5)$					
A	PC[1]	0.4587	0.0962	0.1907	0.4212	0.0812	0.1382	0.4111	0.0590	0.1223
	PC[2]	0.3801	0.1052	0.2233	0.3641	0.0982	0.2190	0.3924	0.0849	0.1796
	PC[3]	0.3899	0.0997	0.2032	0.3702	0.0943	0.1970	0.3870	0.0753	0.1779
B	PC[1]	0.4198	0.0852	0.1631	0.3416	0.0756	0.1214	0.3438	0.0474	0.0894
	PC[2]	0.3926	0.0885	0.1648	0.3159	0.0772	0.1279	0.3273	0.0507	0.0950
	PC[3]	0.3888	0.0936	0.1735	0.3096	0.0794	0.1301	0.3267	0.0531	0.1138
C	PC[1]	0.4489	0.0791	0.1301	0.4436	0.0654	0.1013	0.3844	0.0385	0.0836
	PC[2]	0.3983	0.0830	0.1566	0.3956	0.0739	0.1211	0.3542	0.0440	0.0885
	PC[3]	0.4059	0.0819	0.1391	0.4063	0.0695	0.1097	0.3599	0.0414	0.0872
D	PC[1]	0.4175	0.0613	0.1194	0.4379	0.0582	0.0809	0.3535	0.0348	0.0667
	PC[2]	0.4061	0.0709	0.1204	0.4331	0.0600	0.0850	0.3509	0.0368	0.0691
	PC[3]	0.4031	0.0742	0.1223	0.4337	0.0619	0.0891	0.3461	0.0379	0.0815

Table 6: Interval estimation results of σ1.

Test	Scheme	ACI-NA BCI ACI-NL HPD					
				Prior-1		Prior-2	
		(T_1,T_2) = (2.5,5.5)					
A	PC[1]	1.334	0.904	0.770	0.926	0.763	0.928
		1.706	0.895	0.738	0.928	0.727	0.931
	PC[2]	1.384	0.899	0.877	0.918	0.870	0.920
		1.775	0.890	0.863	0.920	0.850	0.923
	PC[3]	1.350	0.902	0.782	0.924	0.776	0.926
		1.732	0.893	0.762	0.926	0.750	0.929
B	PC[1]	1.100	0.912	0.637	0.932	0.631	0.934
		1.422	0.905	0.621	0.934	0.612	0.937
	PC[2]	1.234	0.909	0.659	0.931	0.653	0.933
		1.589	0.901	0.638	0.933	0.629	0.936
	PC[3]	1.251	0.907	0.679	0.929	0.674	0.931
		1.609	0.898	0.662	0.931	0.652	0.934
C	PC[1]	0.959	0.916	0.572	0.935	0.567	0.937
		1.248	0.912	0.568	0.938	0.559	0.941
	PC[2]	1.084	0.913	0.625	0.932	0.619	0.934
		1.374	0.908	0.613	0.934	0.604	0.937
	PC[3]	1.029	0.914	0.618	0.933	0.613	0.935
		1.275	0.911	0.608	0.935	0.599	0.938
D	PC[1]	0.751	0.921	0.481	0.941	0.477	0.943
		0.905	0.918	0.475	0.943	0.468	0.946
	PC[2]	0.811	0.919	0.497	0.940	0.493	0.942

		1.026	0.917	0.490	0.941	0.483	0.944
	PC[3]	0.904	0.917	0.514	0.938	0.510	0.940
		1.115	0.915	0.509	0.940	0.501	0.943
$(T_1,T_2) = (9.5,7.5)$							
A	PC[1]	1.343	0.902	0.785	0.923	0.779	0.925
		1.718	0.892	0.759	0.925	0.748	0.928
	PC[2]	1.426	0.897	0.927	0.916	0.919	0.918
		1.831	0.887	0.911	0.918	0.897	0.921
	PC[3]	1.414	0.900	0.796	0.922	0.790	0.924
		1.816	0.890	0.774	0.924	0.763	0.927
B	PC[1]	1.175	0.908	0.647	0.930	0.641	0.932
		1.488	0.902	0.634	0.932	0.625	0.935
	PC[2]	1.246	0.906	0.667	0.929	0.661	0.931
		1.492	0.898	0.656	0.931	0.646	0.934
	PC[3]	1.269	0.905	0.702	0.926	0.696	0.928
		1.608	0.895	0.694	0.929	0.684	0.932
C	PC[1]	1.007	0.913	0.608	0.933	0.603	0.935
		1.254	0.909	0.578	0.936	0.569	0.939
	PC[2]	1.150	0.910	0.628	0.931	0.623	0.933
		1.482	0.905	0.620	0.932	0.611	0.935
	PC[3]	1.044	0.912	0.621	0.931	0.616	0.933
		1.295	0.908	0.612	0.933	0.603	0.936
D	PC[1]	0.860	0.919	0.490	0.939	0.486	0.941
		1.085	0.915	0.487	0.941	0.480	0.944
	PC[2]	0.951	0.917	0.504	0.938	0.500	0.940
		1.180	0.914	0.498	0.940	0.490	0.943
	PC[3]	0.964	0.915	0.519	0.936	0.515	0.938
		1.238	0.912	0.514	0.938	0.506	0.941

Table 7: Interval estimation results of σ2.

Test	Scheme	ACI-NA BCI ACI-NL HPD					
				Prior-1		Prior-2	
$(T_1,T_2) = (2.5,5.5)$							
A	PC[1]	1.216	0.911	1.041	0.917	1.029	0.920
		1.245	0.906	1.024	0.919	1.005	0.922
	PC[2]	1.226	0.909	1.099	0.915	1.087	0.918
		1.285	0.904	1.060	0.918	1.041	0.921
	PC[3]	1.257	0.907	1.136	0.913	1.124	0.916
		1.293	0.902	1.095	0.916	1.075	0.919
B	PC[1]	1.019	0.915	0.925	0.921	0.915	0.924
		1.041	0.910	0.821	0.924	0.806	0.927
	PC[2]	1.029	0.914	0.939	0.920	0.928	0.923
		1.053	0.910	0.848	0.923	0.832	0.926
	PC[3]	1.031	0.914	0.954	0.920	0.944	0.923
		1.095	0.909	0.873	0.923	0.857	0.926
C	PC[1]	0.751	0.924	0.725	0.928	0.717	0.931
		0.809	0.919	0.718	0.931	0.705	0.934
	PC[2]	0.801	0.921	0.740	0.926	0.732	0.929
		0.917	0.916	0.734	0.929	0.720	0.932
	PC[3]	0.811	0.920	0.744	0.926	0.736	0.929
		0.919	0.915	0.736	0.929	0.723	0.932
D	PC[1]	0.598	0.932	0.555	0.936	0.549	0.939
		0.601	0.927	0.512	0.939	0.502	0.942
	PC[2]	0.613	0.931	0.606	0.933	0.599	0.936

		0.657	0.925	0.595	0.936	0.584	0.939
	PC[3]	0.656	0.928	0.628	0.931	0.621	0.934
		0.685	0.923	0.610	0.934	0.599	0.937
		(T_1,T_2) = (9.5,7.5)					
A	PC[1]	1.209	0.909	1.063	0.915	1.051	0.918
		1.237	0.905	1.028	0.918	1.009	0.921
	PC[2]	1.237	0.907	1.105	0.913	1.093	0.916
		1.265	0.903	1.066	0.917	1.047	0.920
	PC[3]	1.284	0.905	1.139	0.911	1.126	0.914
		1.308	0.901	1.099	0.915	1.079	0.918
B	PC[1]	1.016	0.913	0.905	0.919	0.895	0.922
		1.038	0.909	0.829	0.923	0.814	0.926
	PC[2]	1.026	0.912	0.926	0.918	0.916	0.921
		1.050	0.908	0.852	0.922	0.836	0.925
	PC[3]	1.028	0.912	0.962	0.917	0.952	0.920
		1.075	0.908	0.881	0.921	0.865	0.924
C	PC[1]	0.727	0.922	0.722	0.926	0.714	0.929
		0.774	0.918	0.715	0.930	0.702	0.933
	PC[2]	0.808	0.919	0.738	0.924	0.729	0.927
		0.913	0.915	0.731	0.928	0.718	0.931
	PC[3]	0.819	0.917	0.775	0.922	0.767	0.925
		0.940	0.914	0.733	0.927	0.720	0.930
D	PC[1]	0.606	0.930	0.585	0.933	0.578	0.936
		0.603	0.925	0.529	0.938	0.520	0.941
	PC[2]	0.625	0.928	0.611	0.932	0.604	0.935
		0.684	0.923	0.608	0.935	0.597	0.938
	PC[3]	0.677	0.926	0.649	0.929	0.642	0.932
		0.707	0.921	0.611	0.933	0.600	0.936

Table 8: Interval estimation results of α.

Test	Scheme	ACI-NA					
		BCI	ACI-NL				
		HPD					
				Prior-1		Prior-2	
		$(T_1,T_2) = (2.5,5.5)$					
A	PC[1]	2.256	0.868	1.725	0.886	1.711	0.890
		2.282	0.864	1.497	0.889	1.482	0.894
	PC[2]	2.145	0.872	1.639	0.890	1.626	0.894
		2.171	0.868	1.418	0.893	1.404	0.898
	PC[3]	1.752	0.878	1.576	0.894	1.563	0.898
		1.786	0.874	1.337	0.897	1.324	0.902
B	PC[1]	1.712	0.880	1.558	0.896	1.546	0.900
		1.726	0.876	1.313	0.899	1.300	0.904
	PC[2]	1.505	0.883	1.485	0.900	1.474	0.905
		1.514	0.879	1.295	0.903	1.282	0.908
	PC[3]	1.498	0.884	1.441	0.903	1.430	0.908
		1.510	0.880	1.261	0.906	1.248	0.911
C	PC[1]	1.424	0.886	1.404	0.906	1.393	0.911
		1.433	0.882	1.254	0.909	1.241	0.914
	PC[2]	1.275	0.890	1.217	0.910	1.207	0.915
		1.361	0.886	1.200	0.912	1.188	0.917
	PC[3]	1.214	0.892	1.187	0.912	1.177	0.917
		1.357	0.888	1.158	0.915	1.147	0.920
D	PC[1]	1.102	0.896	1.040	0.913	1.032	0.918
		1.317	0.892	1.036	0.917	1.026	0.922
	PC[2]	1.074	0.898	0.893	0.916	0.886	0.921
		1.288	0.894	0.887	0.919	0.878	0.924

	PC[3]	0.936	0.901	0.884	0.918	0.877	0.923
		1.084	0.896	0.871	0.921	0.862	0.926
$(T_1,T_2) = (9.5,7.5)$							
A	PC[1]	2.369	0.865	1.797	0.883	1.783	0.888
		2.395	0.861	1.711	0.886	1.694	0.891
	PC[2]	2.242	0.869	1.747	0.887	1.733	0.892
		2.269	0.865	1.516	0.890	1.501	0.895
	PC[3]	1.815	0.875	1.655	0.891	1.642	0.896
		1.829	0.871	1.431	0.894	1.416	0.899
B	PC[1]	1.771	0.877	1.640	0.893	1.627	0.898
		1.785	0.873	1.410	0.896	1.396	0.901
	PC[2]	1.552	0.880	1.544	0.897	1.531	0.902
		1.563	0.876	1.380	0.899	1.366	0.904
	PC[3]	1.468	0.881	1.459	0.900	1.448	0.905
		1.562	0.877	1.360	0.903	1.346	0.908
C	PC[1]	1.370	0.883	1.359	0.903	1.348	0.908
		1.561	0.879	1.301	0.907	1.288	0.912
	PC[2]	1.235	0.887	1.211	0.907	1.201	0.912
		1.543	0.883	1.292	0.909	1.279	0.914
	PC[3]	1.214	0.889	1.199	0.909	1.190	0.914
		1.220	0.885	1.105	0.912	1.094	0.917
D	PC[1]	1.179	0.894	1.171	0.910	1.162	0.915
		1.207	0.889	1.059	0.914	1.049	0.919
	PC[2]	1.086	0.895	1.076	0.913	1.068	0.918
		1.171	0.891	0.990	0.916	0.980	0.921
	PC[3]	1.064	0.898	1.057	0.915	1.049	0.920
		1.162	0.894	0.958	0.918	0.948	0.923

Table 9: Interval estimation results of δ.

Test	Scheme	ACI-NA					
	BCI ACI-NL						
	HPD						
				Prior-1		Prior-2	
$(T_1,T_2) = (2.5,5.5)$							
A	PC[1]	0.248	0.933	0.240	0.936	0.237	0.937
		0.244	0.935	0.209	0.940	0.205	0.941
	PC[2]	0.252	0.932	0.242	0.935	0.239	0.936
		0.246	0.934	0.212	0.939	0.208	0.940
	PC[3]	0.267	0.930	0.250	0.933	0.247	0.934
		0.255	0.932	0.220	0.937	0.216	0.938
B	PC[1]	0.232	0.936	0.223	0.939	0.220	0.940
		0.225	0.938	0.198	0.943	0.195	0.944
	PC[2]	0.237	0.934	0.227	0.937	0.224	0.938
		0.232	0.936	0.202	0.941	0.199	0.942
	PC[3]	0.244	0.933	0.239	0.936	0.236	0.937
		0.242	0.935	0.206	0.940	0.202	0.941
C	PC[1]	0.193	0.941	0.171	0.944	0.169	0.945
		0.172	0.943	0.166	0.948	0.163	0.949
	PC[2]	0.223	0.939	0.179	0.942	0.177	0.943
		0.183	0.941	0.174	0.946	0.171	0.947
	PC[3]	0.228	0.938	0.191	0.941	0.189	0.942
		0.199	0.940	0.186	0.945	0.183	0.946
D	PC[1]	0.166	0.944	0.158	0.946	0.156	0.947
		0.164	0.945	0.141	0.950	0.139	0.951
	PC[2]	0.175	0.943	0.161	0.945	0.159	0.946

		0.166	0.944	0.145	0.949	0.142	0.950
	PC[3]	0.181	0.942	0.167	0.945	0.165	0.946
		0.170	0.943	0.148	0.948	0.145	0.949
$(T_1,T_2) = (9.5,7.5)$							
A	PC[1]	0.254	0.931	0.240	0.934	0.238	0.935
		0.245	0.933	0.217	0.938	0.213	0.939
	PC[2]	0.257	0.930	0.243	0.933	0.240	0.934
		0.248	0.932	0.223	0.937	0.219	0.938
	PC[3]	0.269	0.928	0.256	0.931	0.253	0.932
		0.261	0.930	0.230	0.935	0.226	0.936
B	PC[1]	0.236	0.934	0.226	0.937	0.223	0.938
		0.229	0.935	0.202	0.941	0.198	0.943
	PC[2]	0.237	0.934	0.233	0.935	0.230	0.936
		0.236	0.934	0.207	0.939	0.204	0.940
	PC[3]	0.245	0.932	0.239	0.934	0.236	0.935
		0.243	0.933	0.210	0.938	0.206	0.939
C	PC[1]	0.208	0.939	0.173	0.941	0.171	0.942
		0.175	0.941	0.167	0.946	0.164	0.948
	PC[2]	0.225	0.937	0.179	0.940	0.177	0.941
		0.187	0.939	0.175	0.944	0.172	0.946
	PC[3]	0.231	0.935	0.192	0.939	0.189	0.940
		0.202	0.938	0.186	0.943	0.183	0.945
D	PC[1]	0.169	0.942	0.150	0.945	0.148	0.946
		0.162	0.943	0.143	0.948	0.140	0.950
	PC[2]	0.179	0.941	0.165	0.943	0.163	0.944
		0.168	0.942	0.149	0.947	0.147	0.949
	PC[3]	0.183	0.940	0.169	0.943	0.167	0.945
		0.171	0.941	0.152	0.946	0.150	0.948

7 Light-Emitting Diode Data Analysis

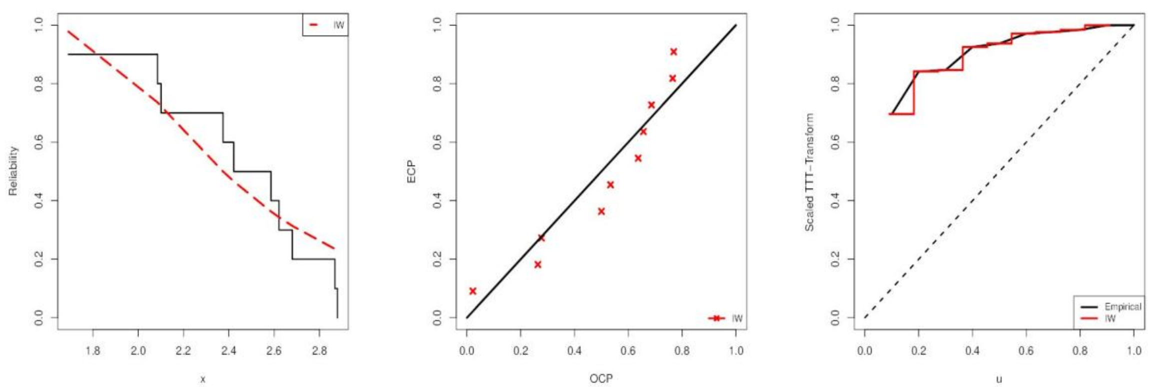
To illustrate how the proposed methodologies can be used in practical situations, this section presents an analysis of an actual dataset from the engineering field representing the lifetime (in hours) of the M00071 white organic light emitting diodes (WOLEDs) mixed with different colors called red, green, and blue under two stress levels, namely: 9.64 and 17.09mA. This data set was originally given by Jianping et al. [12] and rediscussed by Nassar et al. [24] and Nassar et al. [23].

In Table 10, each data point has been divided by a thousand for computational purposes. In this table, we assume the data set at 9.64mA by X (with $r = 10$) and Y (with $k = 10$). To check if the IW distribution is adequate for the WOLED data sets, the MLEs (with their standard errors (SEs)) of IW parameters σ and α are obtained first to build the Kolmogorov-Smirnov (K– S) statistic with its P-value; see Table 10. It shows that the calculated P-value is far from the significance percentage (5%), therefore, the proposed IW distribution fits the WOLED data sets effectively.

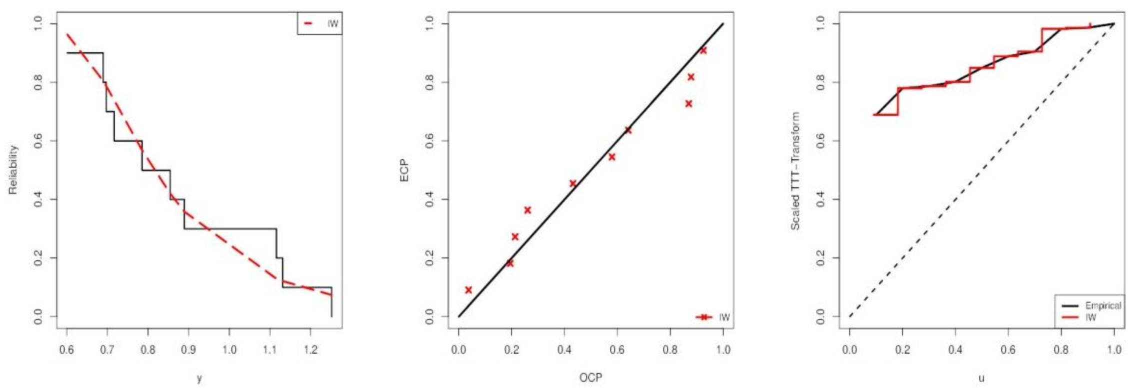
Table 10: Two WOLED data sets.

Data	Lifetimes	MLE(SE)		K–S(P -Value)
		σ	α	
X	1.6915, 2.0847, 2.1003, 2.3745, 2.4215, 2.5860, 2.6215, 2.6805, 2.8680, 2.8795	56.475(41.406)	5.0257(1.0309)	0.2311(0.5824)
Y	0.6015, 0.6897, 0.6973, 0.7165, 0.7855, 0.8545, 0.8895, 1.1157, 1.1313, 1.2515	0.2435(0.1363)	5.1276(1.2892)	0.1703(0.8886)

The estimated/empirical reliability, probability-probability (PP) (known as a visual tool showing the relationship between observed cumulative probability (OCP) and expected cumulative probability (ECP)), and scaled–TTT transform plots based on the WOLED data sets are shown in Figure 1. It shows that the IW model offers an adequate fit for the WOLED data sets. It also points out that the WOLED data sets provide an increasing failure rate. Additionally, Figure 2 shows that the classical estimated values of σ and α developed from Data-X (or Data-Y) exist and are unique. From now on, we suggest utilizing the acquired maximum likelihood estimates of σ and α (reported in Table 10) as starting points to run any additional computations.

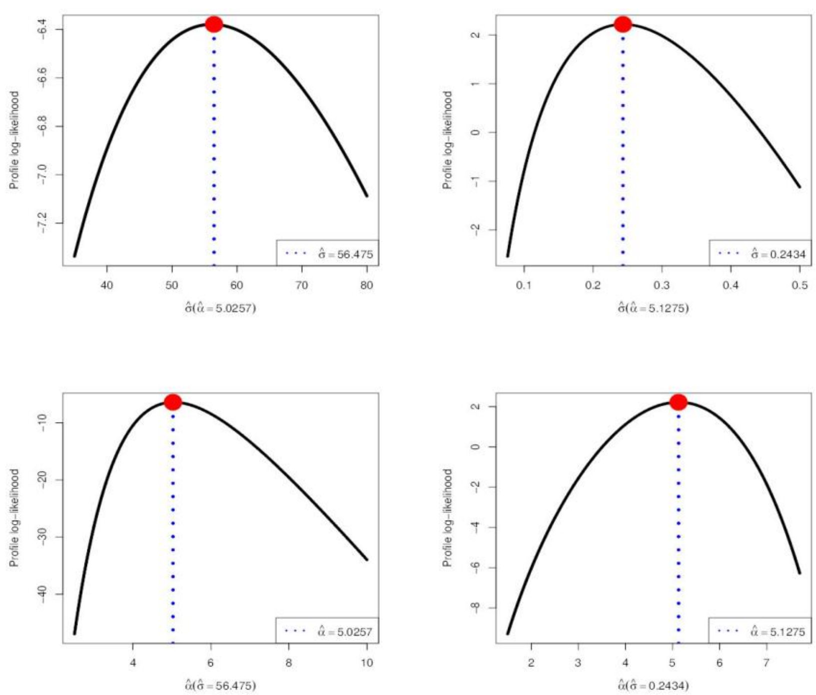


(a) Data-X



(b) Data-Y

Figure 1: The fitted reliability (left), PP (center), and scaled TTT-transform (right) from WOLED data sets.



(a) Data-*X* **(b) Data-*Y***

Figure 2: The profile log-likelihood curves of σ (top) and α (bottom) from WOLED data sets.

Now, from the entire WOLED data sets reported in Table 10, several TII-PHC samples (with $r = m5$) based on different choices of T_i, $i = 1,2$, and progressive designs **R** and **S** are created; see Table 11. Since there is no available prior information, the hyper-parameters a_i and b_i for $i = 1,2,3$, of α and σ_i, $i = 1,2$, are set to be 0.001, although the prior densities for all parameters are considered proper. This setting implies that the prior densities are almost non-informative. Employing the MCMC procedure, we repeated the M-H sampler 50,000 times and then ignored the first 10,000 times as burn-in. As a result, Table 12 lists the point estimates (with their SEs) as well as the interval estimates (with their interval lengths (ILs)) of σ_1, σ_2, α, and δ developed by the maximum likelihood and Bayes procedures. As anticipated from Table 12, the point estimates for all unknown parameters appear to be close to each other. Identical performance is also noted in the case of interval estimates. It is also shown that, in terms of the smallest values of SE and IL, the Bayesian point (or interval) estimates outperform those obtained from likelihood estimates.

Table 11: Three artificial TII-PHC samples from WOLED data sets.

Sample	$\{T_1(D_1),T_2(D_2)\}$	$\{\mathbf{R},\mathbf{S}\}$	$(RD*1,SD*2)$	$\{\underline{x},y\}$
S_1	{2.5(3),0.7(2)}	{(5,0*4),(5,0*4)}	(0,0)	{(1.6915, 2.1003, 2.4215, 2.5860, 2.6805), (0.6015, 0.6973, 0.7855, 0.8545, 1.1157)}
S_2	{2.5(4),0.8(3)}	{(0*2,5,0*2),(0*2,5,0*2)}	(0,0)	{(1.6915, 2.0847, 2.1003, 2.4215, 2.8680), (0.6015, 0.6897, 0.6973, 0.8545,

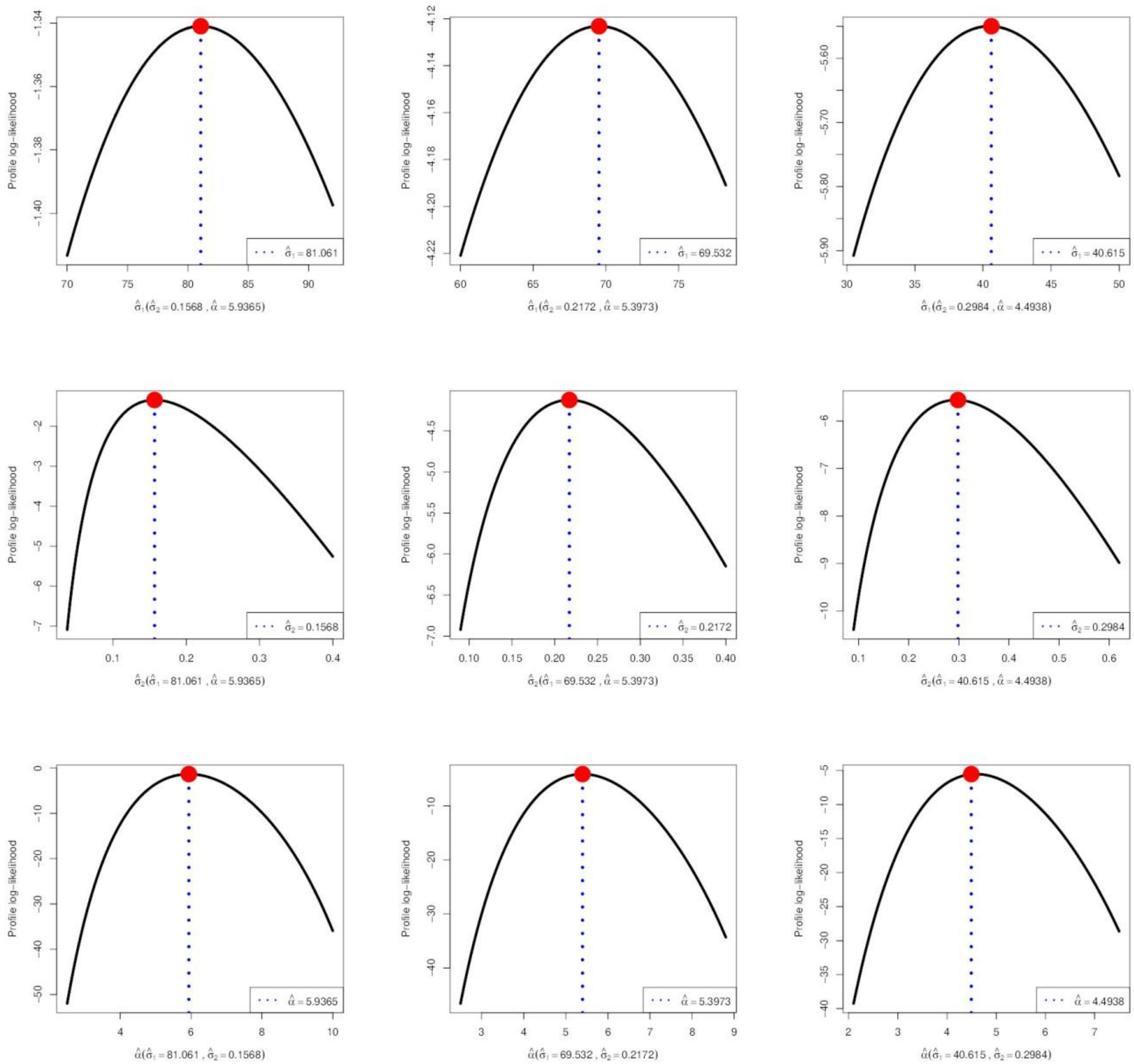
				1.1313)}
S$_3$	{2.6(5),0.8(5)}	{(0*4,5),(0*4,5)}	(5,5)	{(1.6915, 2.0847, 2.1003, 2.3745, 2.4215), (0.6015, 0.6897, 0.6973, 0.7165, 0.7855)}

Table 12: The point and 95% interval estimates of σ1, σ2, α, and δ from WOLED data sets.

Sample	Par.	MLE		MCMC		ACI-NA / ACI-NL			BCI / HPD		
		Est.	SE	Est.	SE	Low.	Upp.	IL	Low.	Upp.	IL
$\mathcal{S}_1$	σ_1	81.061	26.710	81.054	0.0257	28.709	133.41	104.70	81.005	81.103	0.0980
						42.494	154.63	112.13	81.005	81.102	0.0979
	σ_2	0.1568	0.0755	0.1491	0.0240	0.0089	0.3047	0.2958	0.1057	0.1945	0.0888
						0.0611	0.4027	0.3417	0.1049	0.1935	0.0885
	α	5.9365	0.7114	5.9302	0.0260	4.5422	7.3308	2.7886	5.8811	5.9798	0.0987
						4.6939	7.5082	2.8143	5.8810	5.9797	0.0987
	δ	0.9981	0.0013	0.9982	0.0003	0.9955	1.0006	0.0051	0.9976	0.9987	0.0011
						0.9955	1.0006	0.0051	0.9976	0.9987	0.0011
$\mathcal{S}_2$	σ_1	69.532	30.557	69.526	0.0258	9.6408	129.42	119.78	69.476	69.575	0.0985
						29.384	164.54	135.15	69.476	69.575	0.0984
	σ_2	0.2172	0.0954	0.2095	0.0248	0.0301	0.4042	0.3741	0.1639	0.2559	0.0920
						0.0918	0.5138	0.4221	0.1638	0.2557	0.0919
	α	5.3973	0.7234	5.3911	0.0259	3.9795	6.8152	2.8358	5.3419	5.4404	0.0985
						4.1504	7.0189	2.8685	5.3418	5.4403	0.0984
	δ	0.9969	0.0024	0.9970	0.0004	0.9922	1.0015	0.0093	0.9963	0.9976	0.0013
						0.9923	1.0015	0.0093	0.9963	0.9977	0.0013
$\mathcal{S}_3$	σ_1	40.615	15.442	40.210	0.0258	11.951	72.482	60.531	40.161	40.259	0.0983
						20.613	86.464	65.851	40.161	40.259	0.0982
	σ_2	0.2984	0.1171	0.2907	0.0254	0.0689	0.5278	0.4589	0.2438	0.3384	0.0946
						0.1383	0.6438	0.5055	0.2437	0.3381	0.0944
	α	4.4939	0.6097	4.4876	0.0259	3.2988	5.6889	2.3900	4.4384	4.5369	0.0984
						3.4445	5.8628	2.4183	4.4382	4.5365	0.0984
	δ	0.9930	0.0045	0.9932	0.0006	0.9843	1.0017	0.0174	0.9920	0.9943	0.0022
						0.9843	1.0017	0.0174	0.9920	0.9943	0.0022

Figure 3 confirms the existence and uniqueness of the acquired MLEs of $\hat{\sigma}_1$, $\hat{\sigma}_2$, and $\hat{\alpha}$ of σ_1, σ_2, and α, respectively, calculated based on S$_i$ for i =

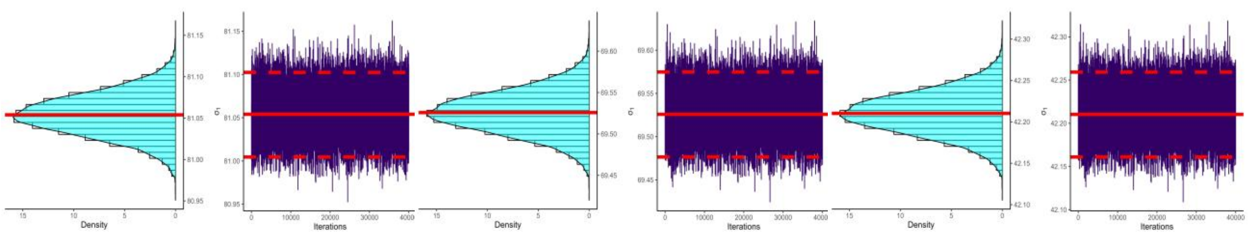
1,2,3. Moreover, Figure 3 shows that all estimates of σ_1, σ_2, or α support all classical point estimation results listed in Table 12.



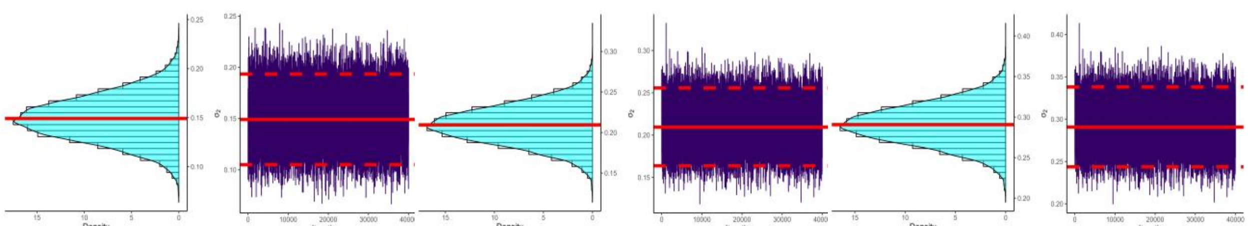
(a) Sample S_1 (b) Sample S_2 (c) Sample S_3

Figure 3: The log-likelihoods of σ_1 (left), σ_2 (middle), and α (right) from WOLED data sets.

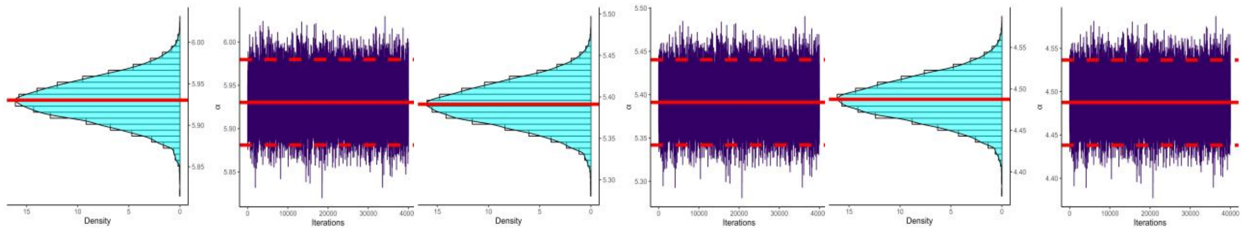
To highlight the convergence of the offered MCMC iterations (after burn-in) of σ_1, σ_2, α, and δ from WOLED data sets, density (along with Gaussian curve) and trace plots are depicted in Figure 4. In each trace plot, for specification, the sample mean and the two HPD credible interval bounds of each unknown parameter are defined by soled (–) and dashed (- - -) lines, respectively. Additionally, the sample mean in each density plot is represented by a soled (–) line. Figure 4 indicates that the MCMC draws yielded from the suggested conditional posterior distributions of σ_1, σ_2, α, or δ converge satisfactorily. It also shows that the burn-in sample has a suitable size to discard the influence of the initial guess points. It also verifies that the densities of all unknown parameters generated by the M-H sampler are fairly-symmetrical.



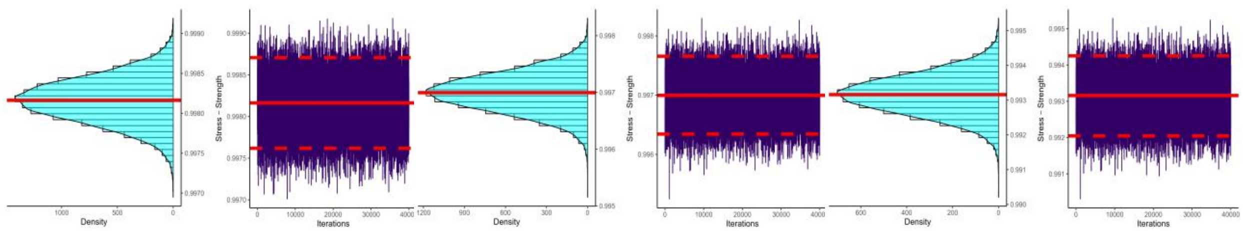
(a) σ_1



(b) σ_2



(c) α



(d) δ

Figure 4: Density (left) and trace (right) of σ_1, σ_2, α, and δ using S_1 (left), S_2 (middle), and S_3 (right) from WOLED data sets.

Again, based on the staying 40,000 MCMC iterations of all unknown parameters, in Table 13, several statistics such as mean, median, mode, 1st quartile (Q_1), 3rd quartile (Q_3), standard deviation (St.D), and skewness (Skew.) of σ_1, σ_2, α, and δ are presented. All results provided in Table 13 support the same findings shown in Table 12.

Table 13: Several statistics of σ_1, σ_2, α, and δ from WOLED data sets.

Sample	Par.	Mean	Mode	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	St.D	Skew.
$\mathcal{S}_1$	σ_1	81.0540	81.0033	81.0372	81.0537	81.0709	0.02494	0.01909
	σ_2	0.14914	0.13912	0.13337	0.14873	0.16439	0.02277	0.09614
	α	5.93024	5.89354	5.91350	5.93007	5.94720	0.02522	-0.01205
	δ	0.99816	0.99829	0.99798	0.99817	0.99836	0.00028	-0.09459
$\mathcal{S}_2$	σ_1	69.5258	69.4752	69.5091	69.5256	69.5428	0.02502	0.02063
	σ_2	0.20948	0.19948	0.19337	0.20919	0.22541	0.02358	0.05869
	α	5.39113	5.35436	5.37433	5.39098	5.40810	0.02517	-0.02248
	δ	0.99700	0.99714	0.99677	0.99700	0.99723	0.00034	-0.05687
$\mathcal{S}_3$	σ_1	40.2103	40.1597	40.1935	40.2101	40.2272	0.02500	0.01654
	σ_2	0.29073	0.28068	0.27427	0.29051	0.30709	0.02423	0.03418
	α	4.48756	4.45087	4.47079	4.48739	4.50447	0.02512	-0.01861
	δ	0.99316	0.99339	0.99278	0.99316	0.99354	0.00057	-0.03130

8 CONCLUSION

In this paper, under the TII-PHC scheme, when the stress and strength follow two independent IWDs, Different methods for estimating the stress–strength parameter are applied. When the common parameter is unknown, it is observed that the MLEs of the three unknown parameters can be obtained by solving one nonlinear equation. Also, the asymptotic distribution of δ was found which was used to compute the asymptotic confidence intervals. The Bayes estimate of δ by using the MCMC method, We achieve the highest posterior density (HPD) credible intervals. In general, it is clear from the simulation results that the proposed point estimations from the Bayes method behave better compared to those created from the maximum likelihood method and the simulation results that the proposed interval estimations by the BCI (or HPD) method outperformed those created by the ACI-NA (or ACI-NL) method. Lastly, the Metropolis-Hastings algorithm is recommended to estimate the unknown parameters when the stress and strength follow two independent IW populations when the sample is gathered from the proposed TII-PHC strategy. A real data example to Light-Emitting Diodes is also discussed for illustration purposes.

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