The effect of Retrial Rate on the *M/G/*1 Retrial Queueing System

Dr. Ibrahim Mousa Abdelfatah

Professor of Statistics

Department of Statistics and Insurance

Faculty of Commerce

Zagazig University

Mohammed Hassan Hassan

Demonstrator

Department of Statistics and Insurance

Faculty of Commerce

Zagazig University

Abstract

In this paper, we consider the main model of *M/G/I* retrial queueing system. We will introduce the previous model with some important properties that affect the retrial system state. Also, the joint distribution of the server state and the length of queue in the steady state and the limit theorems for the stationary distribution of the queue length will be introduced. A simulation models will be used to indicate the difference between the model with low rate of retrials and the model with high rate of retrials.

الملخص

نتناول فى هذا البحث إحدى نماذج صفوف الإنتظار مع إعادة المحاولة وهو النموذج (*M/G/I*)، ويحتوى هذا النموذج على مركز خدمة وحيد وفيه يصل العملاء للحصول على الخدمة المطلوبة ولكنهم لا يحصلون على الخدمة فى المحاولة الأولى لأنهم يجدون مركز الخدمة مشغول بخدمة عملاء آخرين. ومن ثم فهم يغادرون مركز الخدمة بسبب طول صف الإنتظار أو أنهم لا يتمتعون بالصبر ولديهم بعض الأشياء الآخرى يريدوا أن ينجزوها، ولكنهم بعد بعض الوقت العشوائى يعودون مرة آخرى إلى مركز الخدمة للحصول على الخدمة ومن ثم يغادرون مركز الخدمة. بعض الخصائص الخاصة بالنموذج المذكور سيتم تناولها؛ مثل: التوزيع المشترك لوضع مركز الخدمة وطول صف الإنتظار فى حالة الإستقرار. تم تطبيق أسلوب المحاكاة لتوضيح ودراسة بعض الخصائص للنموذج المذكور أعلاه؛ مثل: عدد العملاء التى تمت خدمتهم، متوسط وقت الخدمة وكذلك متوسط وقت إعادة المحاولة.

1. Introduction

Retrial queues can be defined as clients come to the system and if they find all servers occupied, they will make another attempt for having the service again after some time. Retrial queues have been broadly used to display numerous issues in phone systems, call centers, telecommunication networks, personal computers (PC) networks and PC systems, and in everyday life.

The general queuing system with retrials may be described as follows: If there is an available server when a client comes to the system, this client will get the service immediately and will depart the system after he receives the service. On the other hand, any client who finds all servers occupied upon arrival joins a retrial group, called an orbit, and then attempts to be served after a random time. If there is an available server when a client from the orbit attempts to be served, this client receives service instantly and leaves the system after the service completion. Otherwise the client comes back to the orbit immediately and repeats the retrial process.

2. Description of the main model of *M/G/I* type

The *M/G/*1 retrial queue model can be defined as clients arrive to the system by a Poisson arrival process of rate λ. The service times are independent and identically distributed (i.i.d) random variables with a general random variable S. Let $\beta(s) = E[e^{-sS}]$, $s \geq 0$, be the Laplace–Stieltjes transform (LST) of the service time distribution, and $\beta^{(k)}$ be the kth moment of the service time, i.e., $\beta^{(k)} = E[S^k] = (-1)^k \frac{d^k}{ds^k}\beta(s)_{s=0+}$ .The retrial time, i.e., the time interval between two consecutive attempts made by a client in the

orbit, is exponentially distributed with mean ν^{-1}. The arrival process, the service times, and the retrial times are assumed to be mutually independent. The traffic load ρ is defined as $\rho = \lambda\beta^{(1)}$. It is assumed that $\rho < 1$ for stability of the system.

3. Joint distribution of the server state and the queue length in the steady state

When $\rho < 1$, consider now that the general distribution function *B(x)* of the service times.

For the model *M/G/1* retrial queue in the steady state, the joint distribution of the server state and the length of queue is given by

$$P_{0n} = \mathrm{P}\{C(t) = 0, N(t) = n\}, \quad (1)$$

$$p_{1n}(x) \quad = \quad \frac{d}{dx}\mathrm{P}\{C(t) = 1, \xi(t) < x, N(t) = n\} \quad (2)$$

The partial generating functions are given by

$$\begin{aligned} p_0(z) \quad &\equiv \quad \sum_{n=0}^{\infty} z^n p_{0n} \\ &= \quad (1-\rho)\exp\left\{\frac{\lambda}{\mu}\int_1^z \frac{1-k(u)}{k(u)-u}du\right\} \end{aligned}, \quad (3)$$

$$p_1(z,x) \equiv \sum_{n=0}^{\infty} z^n p_{1n}(x)$$

$$= \lambda \frac{1-z}{k(z)-z} p_0(z)[1-B(x)]e^{-(\lambda-\lambda z)x} \quad \textbf{(4)}$$

If in the case *C* (t) = 1 we neglect the elapsed service time $\xi(t)$, then for the probabilities P_{1n} = P{ *C* (t) = 1, *N* (t) = n} we have

$$p_1(z) \equiv \sum_{n=0}^{\infty} z^n p_{1n} = \frac{1-k(z)}{k(z)-z} p_0(z). \quad \textbf{(5)}$$

4. Limit theorems for the stationary distribution of the queue length

Despite of the characteristics of the performance for the system under consideration are available in explicit form, they are cumbersome.

However in some domains of the system parameters, the distribution q_n can be approximated by standard distributions such as Gaussian distribution or the gamma distribution. For that we will investigate the asymptotic behavior of the queue length under limit values of various parameters.

4.1 Heavy traffic

For the case of heavy traffic when arrival rate λ increases in such a way that $\rho \rightarrow$ 1- 0.

If the *M/G/1* type retrial queue is in the steady state and $\beta_2 < \infty$ then

$$\lim_{\lambda\to 1/\beta_1 -0} \mathrm{E}e^{-s(1-\rho)N(t)} = \left(1+\frac{\beta_2}{2\beta_1^2}s\right)^{-1-\frac{2\beta_1}{\mu\beta_2}}, \quad (6)$$

i.e. under heavy traffic queue length *N* (t) asymptotically has a gamma distribution.

4.2 Low rate of retrials

We can describe the queue length distribution in the case of low rate of retrials when $\mu \to 0$ as following

If $\beta_2 < \infty$ then as $\mu \longrightarrow 0$ the queue length is asymptotically Gaussian with

$\frac{\lambda\rho}{(1-\rho)\mu}$ **mean** $\frac{\lambda^3\beta_2+2\lambda\rho-2\lambda\rho^2}{2(1-\rho)^2\mu}$ **and variance .**

4.3 High rate of retrials

In our real situations clients will repeat their calls practically instantly, so the rate of retrials will be high. Then, the length of queue distribution when the rate of retrials is high will be considered when $\mu \longrightarrow \infty$.

5. Numerical Results

Arena simulation software v.14 will be used to build a simulated model for the model with low rate of retrials and the model with high rate of retrials. Then, we can study the qualitative and quantitative behavior of the two models. For every model the program will generate data and then construct the basic model for every case separately, then the results of the two models will be analyzed and we will make a comparison between the results of the two models.

The following shape indicates the model of M/G/1 with retrial

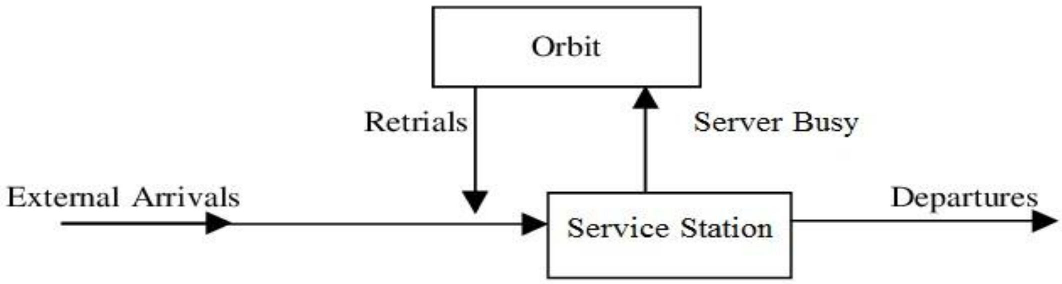

Fig (1)

and the system model will be in the Arena software before running it as follows

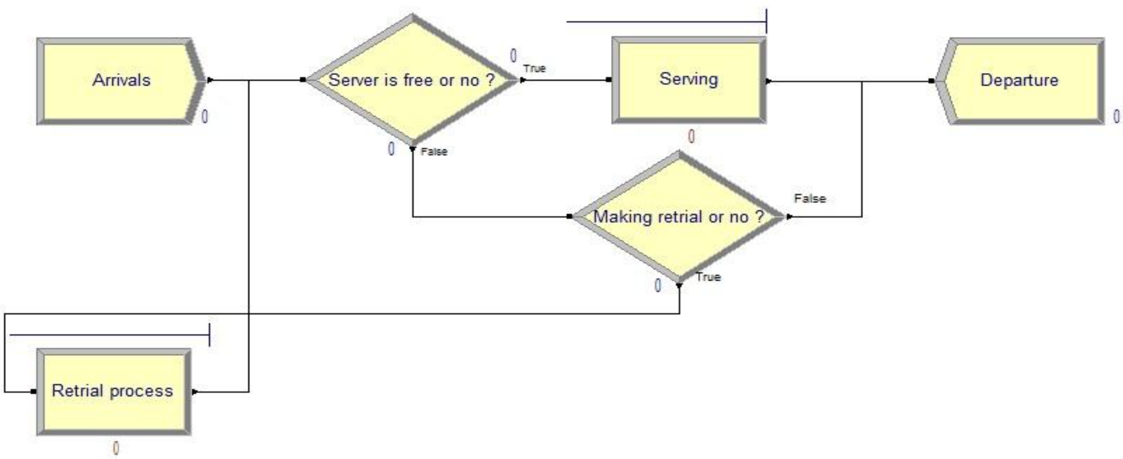

Fig (2)

5.1 The model with low rate of retrials

The obtained results after running the Arena simulation software in different cases can be depicted in the following two tables

Table (1)

System		Arrival rate	Clients served	Retrial in	Retrial out leave	Retrial out retry
System 1		$\lambda = 0.5$	548	477	452	25
μ=1	v=1					
System 2		$\lambda = 0.5$	585	435	415	20
μ=1	v=2					
System 3		$\lambda = 0.5$	288	758	712	46
μ=2	v=1					
System 4		$\lambda = 1$	509	519	491	28
μ=2	v=2					
System 5		$\lambda = 1$	352	687	648	39
μ=3	v=2					
System 6		$\lambda = 1$	521	509	479	30
μ=2	v=3					

Table (2)

System		Arrival Rate	AVG Serving Time	AVG Retrial Time	AVG Client Total Time
System 1		$\lambda = 0.5$	0.1301	0.0112	0.0811
μ=1	v=1				
System 2		$\lambda = 0.5$	0.1213	0.0214	0.0835
μ=1	v=2				
System 3		$\lambda = 0.5$	0.2821	0.0120	0.0924
μ=2	v=1				
System 4		$\lambda = 1$	0.2913	0.0103	0.1731
μ=2	v=2				
System 5		$\lambda = 1$	0.4511	0.0102	0.1823
μ=3	v=2				
System 6		$\lambda = 1$	0.2805	0.0201	0.1713
μ=2	v=3				

As we mentioned before the client come to the system by a process of Poisson with rate λ, if he discovers that the server is free, he will receive the service instantly and the service time has an exponential distribution with parameter μ. On the other hand, the client who discovers that the server is busy will be obliged to depart the service area, but he may retry again after an exponential time with parameter v or depart the system without getting the service.

When we use Arena software, we consider two cases, the first one is that we have fixed the arrival process rate λ at ($\lambda = 0.5$) for the first three systems and the second one is that we have fixed the arrival process rate λ at ($\lambda = 1$) for the last three systems. The previous two cases are performed under low rate of retrials and the clients' number that enter the system fixed at 1000 clients.

The previous results indicate that at the same arrival rate ($\lambda = 0.5$) the number of clients served and the average retrial time have been increased in system (2) when the retrial time rate (v) has been changed from (1 to 2), but the number of clients that has left the system without being served and the average serving time have been decreased. As well in system (3) when the service time rate (μ) has been changed from (1 to 2), the average serving time, the number of clients that have success to being served after retrial and the number of clients that has left the system without being served have been increased to the double almost, but the number of clients served has been decreased to the half approximately.

When the arrival process rate λ has been increased to 1 in the last three systems, the service time rate (μ) and the retrial time rate (v) have been increased to 2 in system (4), we noted that the number of clients served only has been decreased but all other factors have been increased unless the average retrial time has remained the same. We also noted in the last three systems at the same arrival process rate that all factors in case (2) have been changed as well in case (1) but in the opposite way.

5.2 The model with high rate of retrials

The obtained results after running the Arena simulation software in different cases at high rate of retrials can be depicted in the following two tables

Table (3)

System		Arrival rate	Clients served	Retrial in	Retrial out leave	Retrial out retry
System 1		λ = 0.5	479	797	521	276
μ=1	v=1					
System 2		λ = 0.5	522	713	478	235
μ=1	v=2					
System 3		λ = 0.5	218	1152	782	370
μ=2	v=1					
System 4		λ = 1	527	691	473	218
μ=2	v=2					
System 5		λ = 1	350	975	650	325
μ=3	v=2					
System 6		λ = 1	561	669	439	230
μ=2	v=3					

Table (4)

System		Arrival Rate	AVG Serving Time	AVG Retrial Time	AVG Client Total Time
System 1		$\lambda = 0.5$	0.1413	0.0415	0.0914
μ=1	v=1				
System 2		$\lambda = 0.5$	0.1379	0.4372	0.1913
μ=1	v=2				
System 3		$\lambda = 0.5$	0.3443	0.0574	0.1105
μ=2	v=1				
System 4		$\lambda = 1$	0.2855	.0333	0.1822
μ=2	v=2				
System 5		$\lambda = 1$	0.4547	.0471	0.2018
μ=3	v=2				
System 6		$\lambda = 1$	0.2717	0.0850	0.2003
μ=2	v=3				

At the same arrival rate ($\lambda = 0.5$) the number of clients served and the average retrial time have been increased in system (2) when the retrial time rate (v) has been changed from (1 to 2), but the number of clients that has left the system without being served and the average serving time have been decreased. As well in system (3) when the service time rate (μ) has been changed from (1 to 2), the average serving time, the number of clients that have success to being served after retrial and the number of clients that has left the system without being served have been increased, but the number of clients that have been served has been decreased.

5.3 Comparison between results of the model *M/G/1* with low retrial rate and with high retrial rate

The difference between the model M/G/1 with low retrial rate and with high retrial rate can be showed in the following two diagrams

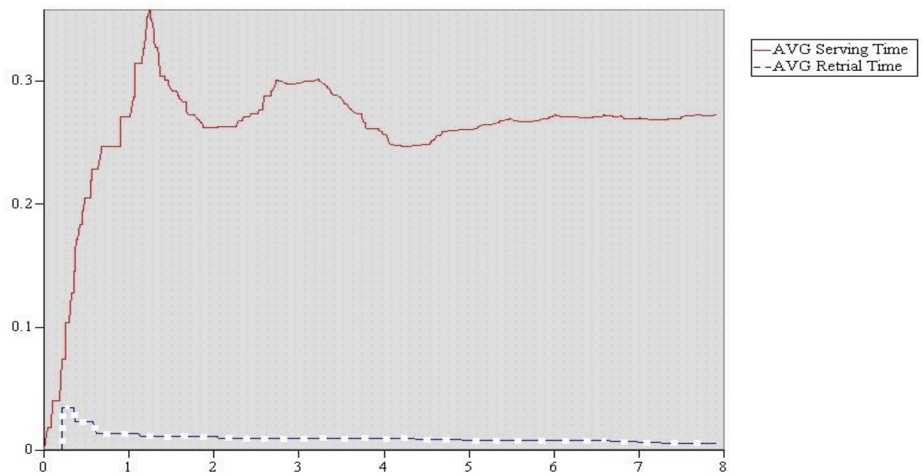

Fig (3) Low Rate of Retrials

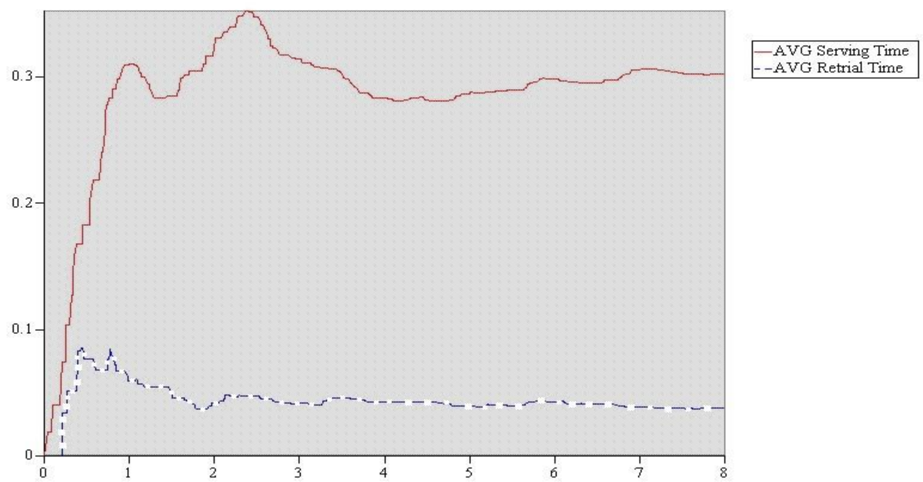

Fig (4) High Rate of Retrials

6. Conclusion

It is clear from the numerical study that the average service times have been changed by a small rate in the two cases, but the average retrial times have been changed by a large rate, where the average retrial times in the case of low rate of retrials have decreased to the half almost.

References

Artalejo, J., and Corral, A. (2007). Waiting time analysis of the *M/G/1* queue with finite retrial group. Naval Research Logistics, 54, 524–529.

Artalejo, J., and Corral, A. (2008). Retrial Queueing Systems. Berlin: Springer.

Artalejo, J., and Phung-Duc, T. (2013). Single server retrial queues with two way communication. Applied Mathematical Modelling, 37, 1811–1822.

Avrachenkov, K., Nain, P., and Yechiali, U. (2014). A retrial system with two input streams and two orbit queues. Queueing Systems, 77, 1–31.

Ayyappan, G., Ganapathi, A. and Sekar, G. (2010). Retrial queuing system with single working vacation under pre-emptive priority service, International Journal of Computer Applications, Vol. 2, No. 2.

Djellab, N. (2006). On the single server retrial queue, Yugoslav Journal of Operations Research, 16, 45-53.

Kim, B., and Kim, J. (2015b). Analysis of the $M^X/G/1$ retrial queue. Annals of Operations Research. doi:10. 1007/s10479-015-1921-6.

Kim, B., and Kim, J. (2015c). Waiting time distributions in an *M/G/1* retrial queue with two classes of customers. Annals of Operations Research. doi:10.1007/s10479-015-1979-1.

Kim, C., Klimenok, V. and Dudin, A. (2012). A G/M/1 retrial queue with constant retrial rate, Sociedad de Estadística e Investigación Operativa, 22, 509-529.

Yamamuro, K. (2012). The queue length in an *M/G/1* batch arrival retrial queue. Queueing Systems, 70, 187–205.